

# Curvilinear Coordinates

The Cartesian coordinates  $x_i$  ( $= x, y, z$ ). For example the spherical polar and cylindrical polar coordinates are two important examples of what are called **curvilinear coordinates**.

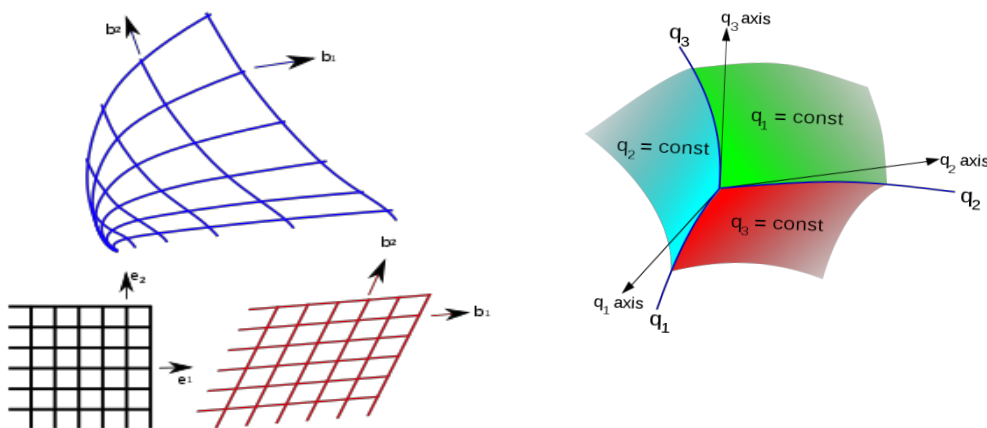
Suppose we change from the Cartesian coordinates  $(x_1, x_2, x_3)$  to the curvilinear coordinates, which we denote  $u_i$ , each of which are functions of the  $x_i$ :-

$$u_1 = u_1(x_1, x_2, x_3), u_2 = u_2(x_1, x_2, x_3) \text{ and } u_3 = u_3(x_1, x_2, x_3)$$

The  $u_i$  should be single-valued, except possibly at certain points, so the reverse transformation,

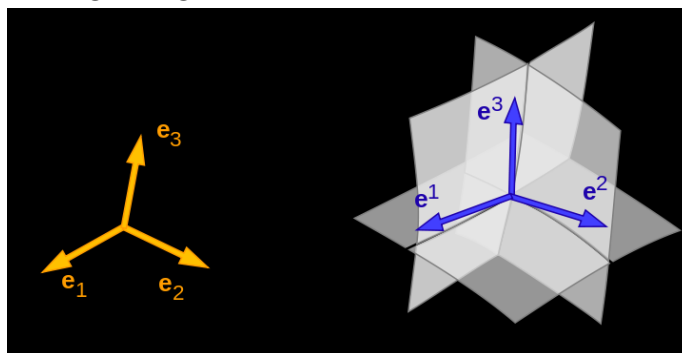
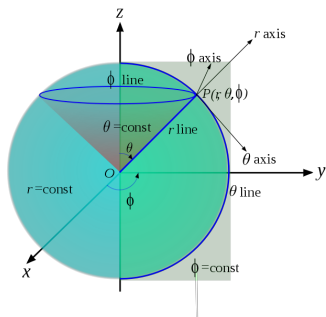
$$x_i = x_i(u_1, u_2, u_3)$$

can be made. A point may be referred to by its Cartesian coordinates  $x_i$ , or by its curvilinear coordinates  $u_i$ . For example, in 2-D, we might have:-

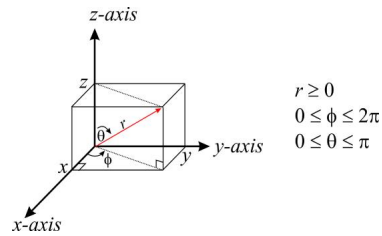


Now consider **coordinate surfaces** defined by keeping one coordinate constant.

- The Cartesian coordinate surfaces ' $x_i = \text{constant}$ ' ( **X=constant 'yz plane'** , **Y=const. 'xz plane'** and **Z=const. 'xy-plane'** with **constant** unit normal vectors  $e_i$  (or  $e_1, e_2$  and  $e_3$ ), ( **i, j, k**) intersecting at right angles.
- The surfaces ' $u_i = \text{constant}$ ' do **not**, in general, have constant unit normal vectors, nor in general do they intersect at right angles.



**Example:** Spherical polar co-ordinates



$$r = \sqrt{x^2 + y^2 + z^2}; \theta = \cos^{-1} \left\{ \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\} ; \phi = \tan^{-1} \left( \frac{y}{x} \right) .$$

The surfaces of constant  $r$ ,  $\theta$ , and  $\phi$  are:-

$r = \text{constant} \implies$  spheres centred at the origin unit normal  $\underline{e}_r$

$\theta = \text{constant} \implies$  cones of semi-angle  $\theta$  and axis along the  $z$ -axis unit normal  $\underline{e}_\theta$

$\phi = \text{constant} \implies$  planes passing through the  $z$ -axis unit normal  $\underline{e}_\phi$

These surfaces are **not** all planes, but they **do** intersect at right angles.

If the coordinate surfaces intersect at right angles (*i.e. the unit normals intersect at right angles*), as in the example of spherical polars, the curvilinear coordinates are said to be **orthogonal**.

1. Orthogonal Curvilinear Coordinates

**Unit Vectors and Scale Factors**

Suppose the point  $P$  has position  $\underline{r} = \underline{r}(u_1, u_2, u_3)$ . If we change  $u_1$  by a small amount,  $du_1$ , then  $\underline{r}$  moves to position  $(\underline{r} + \underline{dr})$ , where

$$\underline{dr} = \frac{\partial \underline{r}}{\partial u_1} du_1 \equiv h_1 \underline{e}_1 du_1$$

where we have defined the **unit vector**  $\underline{e}_1$  and the **scale factor**  $h_1$  by

$$h_1 = \left| \frac{\partial \underline{r}}{\partial u_1} \right| \quad \text{and} \quad \underline{e}_1 = \frac{1}{h_1} \frac{\partial \underline{r}}{\partial u_1} .$$

- The vector  $\underline{e}_1$  is a unit vector in the **direction of increasing**  $u_1$ .
- The scale factor (( **Lami factor**)  $h_1$ ) gives the **magnitude** of  $\underline{dr}$  when we make the change  $u_1 \rightarrow u_1 + du_1$ . Thus for an infinitesimal change of  $u_1$

$$|\underline{dr}| = h_1 du_1$$

Similarly, we can define  $h_i$  and  $e_i$  for  $i = 2$  and  $3$ .

- The unit vectors  $\underline{e}_i$  are **not constant vectors**. In general they are **non-Cartesian** basis vectors, they depend on the position vector  $\underline{r}$ , *i.e.* their directions change as the  $u_i$  are varied.
- If  $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ , then the  $u_i$  are **orthogonal curvilinear coordinates**.

For Cartesian coordinates, the scale factors are unity and the unit vectors  $\underline{e}_i$  reduce to the Cartesian basis vectors we have used throughout the course:

$$\underline{r} = x \underline{e}_1 + y \underline{e}_2 + z \underline{e}_3 \quad \text{so that} \quad h_1 \underline{e}_1 = \frac{\partial \underline{r}}{\partial x} = \underline{e}_1, \text{ etc.}$$

**Example: spherical polars:**  $u_1 = r$ ,  $u_2 = \theta$  and  $u_3 = \phi$  in that order:

$$\underline{r} = r \sin \theta \cos \phi \underline{i} + r \sin \theta \sin \phi \underline{j} + r \cos \theta \underline{k},$$

where to avoid confusion in this section we use  $\underline{i}, \underline{j}, \underline{k}$  for the *Cartesian* basis vectors. (By this stage there should be no confusion with the suffices  $i, j, k$ ). Therefore

$$\begin{aligned} \frac{\partial \underline{r}}{\partial r} &= \sin \theta \cos \phi \underline{i} + \sin \theta \sin \phi \underline{j} + \cos \theta \underline{k} && \implies h_r = 1 \\ \frac{\partial \underline{r}}{\partial \theta} &= r \cos \theta \cos \phi \underline{i} + r \cos \theta \sin \phi \underline{j} - r \sin \theta \underline{k} && \implies h_\theta = r \\ \frac{\partial \underline{r}}{\partial \phi} &= -r \sin \theta \sin \phi \underline{i} + r \sin \theta \cos \phi \underline{j} && \implies h_\phi = r \sin \theta \end{aligned}$$

Thus

$$\begin{aligned} \underline{e}_r &= \sin \theta \cos \phi \underline{i} + \sin \theta \sin \phi \underline{j} + \cos \theta \underline{k} = \underline{r} / r \\ \underline{e}_\theta &= \cos \theta \cos \phi \underline{i} + \cos \theta \sin \phi \underline{j} - \sin \theta \underline{k} \\ \underline{e}_\phi &= -\sin \phi \underline{i} + \cos \phi \underline{j} \end{aligned}$$

These unit vectors are normal to the level surfaces described above (spheres, cones and planes) and are clearly orthogonal:  $\underline{e}_r \cdot \underline{e}_\theta = \underline{e}_r \cdot \underline{e}_\phi = \underline{e}_\theta \cdot \underline{e}_\phi = 0$  and form a RH orthonormal basis:  $\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi$ ,  $\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r$ ,  $\underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta$ .

**Example: Cylindrical polars:**  $u_1 = \rho$ ,  $u_2 = \phi$  and  $u_3 = z$  in that order:

$$\underline{r} = \rho \cos \phi \underline{i} + \rho \sin \phi \underline{j} + z \underline{k}.$$

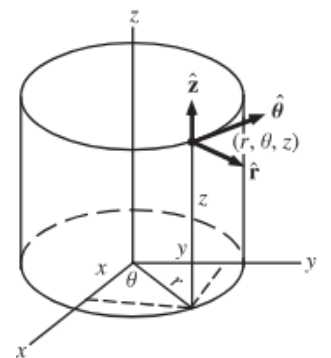
Thus  $\frac{\partial \underline{r}}{\partial \rho} = \cos \phi \underline{i} + \sin \phi \underline{j}$  ;  $\frac{\partial \underline{r}}{\partial \phi} = -\rho \sin \phi \underline{i} + \rho \cos \phi \underline{j}$  ;  $\frac{\partial \underline{r}}{\partial z} = \underline{k}$

and

$$h_\rho = 1 \quad ; \quad h_\phi = \rho \quad ; \quad h_z = 1$$

Therefore

$$\underline{e}_\rho = \cos \phi \underline{i} + \sin \phi \underline{j} ; \underline{e}_\phi = -\sin \phi \underline{i} + \cos \phi \underline{j} ; \underline{e}_z = \underline{k}$$



These unit vectors are normal to the level surfaces described by cylinders about the  $z$ -axis ( $\rho = \text{constant}$ ), planes through the  $z$ -axis ( $\phi = \text{constant}$ ), planes perpendicular to the  $z$  axis ( $z = \text{constant}$ ) and are clearly orthogonal.

**Remark:** An example of a curvilinear coordinate system which is **not** orthogonal is provided by the system of elliptical cylindrical coordinates (see tutorial 9.4).

$$\underline{r} = a \rho \cos \theta \underline{i} + b \rho \sin \theta \underline{j} + z \underline{k} \quad (a \neq b)$$

In the following we shall only consider orthogonal systems

### Arc Length

The arc length  $ds$  is the length of the infinitesimal vector  $\underline{dr}$  :-

$$(ds)^2 = \underline{dr} \cdot \underline{dr} .$$

In Cartesian coordinates

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 .$$

In curvilinear coordinates, if we change all three coordinates  $u_i$  by infinitesimal amounts  $du_i$ , then we have

$$\begin{aligned} \underline{dr} &= \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3 \\ &= h_1 du_1 \underline{e}_1 + h_2 du_2 \underline{e}_2 + h_3 du_3 \underline{e}_3 \end{aligned}$$

For the case of orthogonal curvilinears, because the basis vectors are orthonormal we have

$$(ds)^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

For spherical polars, we showed that

$$h_r = 1, \quad h_\theta = r, \quad \text{and} \quad h_\phi = r \sin \theta$$

therefore

$$(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2$$

## 2. Elements of Area and Volume

Basically we just repeat using scale factors what we did in lectures 18 and 19.

### Vector Area

If  $u_1 \rightarrow u_1 + du_1$ , then  $\underline{r} \rightarrow \underline{r} + \underline{dr}_1$ , where  $\underline{dr}_1 = h_1 \underline{e}_1 du_1$ , and if  $u_2 \rightarrow u_2 + du_2$ , then  $\underline{r} \rightarrow \underline{r} + \underline{dr}_2$ , where  $\underline{dr}_2 = h_2 \underline{e}_2 du_2$ . (Curvature greatly exaggerated in figure!)

On the surface of constant  $u_1$  the **vector area** bounded by  $\underline{dr}_2$  and  $\underline{dr}_3$  is given by

$$\underline{dS}_1 = (\underline{dr}_2) \times (\underline{dr}_3) = (h_2 du_2 \underline{e}_2) \times (h_3 du_3 \underline{e}_3) = h_2 h_3 du_2 du_3 \underline{e}_1,$$

since  $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$  for orthogonal systems.

Thus  $\underline{dS}_1$  is a vector pointing in the direction of the **normal** to the surfaces ' $u_1 = \text{constant}$ ', its magnitude being the area of the small parallelogram with edges  $\underline{dr}_2$  and  $\underline{dr}_3$ . Similarly, one can define  $\underline{dS}_2$  and  $\underline{dS}_3$ .

For the case of **spherical polars**, if we vary  $\theta$  and  $\phi$ , keeping  $r$  fixed, then

$$\underline{dS}_r = h_\theta d\theta \underline{e}_\theta \times (h_\phi d\phi \underline{e}_\phi) = h_\theta h_\phi d\theta d\phi \underline{e}_r = r^2 \sin \theta d\theta d\phi \underline{e}_r.$$

for  $\underline{dS}_\theta$  and  $\underline{dS}_\phi$ .

### Volume

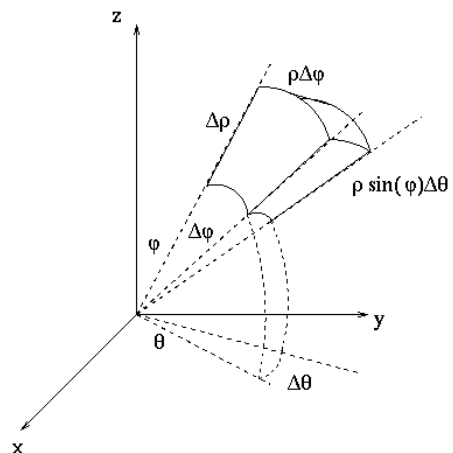
The volume contained in the parallelepiped with edges  $\underline{dr}_1$ ,  $\underline{dr}_2$  and  $\underline{dr}_3$ , is

$$\begin{aligned} dV &= \underline{dr}_1 \cdot \underline{dr}_2 \times \underline{dr}_3 \\ &= (h_1 du_1 \underline{e}_1) \cdot (h_2 du_2 \underline{e}_2) \times (h_3 du_3 \underline{e}_3) \\ &= h_1 h_2 h_3 du_1 du_2 du_3 \end{aligned}$$

because  $\underline{e}_1 \cdot \underline{e}_2 \times \underline{e}_3 = 1$ .

For **spherical polars**, we have

$$dV = h_r h_\theta h_\phi dr d\theta d\phi = r^2 \sin \theta dr d\theta d\phi$$



### Components of a Vector Field in Curvilinear Coordinates

A vector field  $\underline{A}(\underline{r})$  can be expressed in terms of **curvilinear components**  $A_i$ , defined as:–

$$\underline{A}(\underline{r}) = \sum_i A_i(u_1, u_2, u_3) \underline{e}_i$$

where  $\underline{e}_i$  is the  $i$ th basis vector for the curvilinear coordinate system.

For **orthogonal** curvilinear coordinates, the component  $A_i$  is obtained by taking the scalar product of  $\underline{A}$  with the  $i$ th (curvilinear) basis vector  $\underline{e}_i$

$$A_i = \underline{e}_i \cdot \underline{A}(\underline{r})$$

NB  $A_i$  must be expressed in terms of  $u_i$  (not  $x, y, z$ ) when working in the  $u_i$  basis.

**Example** If  $\underline{A} = \underline{i}$  in Cartesian coordinates, then in spherical polars,  $A_r = \underline{A} \cdot \underline{e}_r = \sin \theta \cos \phi$ , etc.

$$\underline{A}(r, \theta, \phi) = \sin \theta \cos \phi \underline{e}_r + \cos \theta \cos \phi \underline{e}_\theta - \sin \phi \underline{e}_\phi.$$

**Example** If  $\underline{A} = \underline{r}$  then in cylindrical polars  $A_\rho = \underline{A} \cdot \underline{e}_\rho = \rho \cos^2 \phi + \rho \sin^2 \phi = \rho$  etc, and

$$\underline{A}(\rho, \phi, z) = \underline{r} = \rho \underline{e}_\rho + z \underline{e}_z$$

### 3. Grad, Div, Curl, and the Laplacian in Orthogonal Curvilinears

We defined the vector operators grad, div, curl firstly in Cartesian coordinates, then most generally through integral definitions without regard to a coordinate system. Here we complete the picture by providing the definitions in any orthogonal curvilinear coordinate system.

#### Gradient

In section (2) we defined the gradient in terms of the change in a scalar field  $f$  when we let  $\underline{r} \rightarrow \underline{r} + \underline{dr}$

$$\delta f = \underline{\nabla} f(\underline{r}) \cdot \underline{dr} \quad (1)$$

Now consider what happens when we write  $f$  in terms of orthogonal curvilinear coordinates  $f = f(u_1, u_2, u_3)$ . As before, we denote the curvilinear basis vectors by  $\underline{e}_1, \underline{e}_2$ , and  $\underline{e}_3$ .

Let  $u_1 \rightarrow u_1 + du_1$ ,  $u_2 \rightarrow u_2 + du_2$ , and  $u_3 \rightarrow u_3 + du_3$ .

By Taylor's theorem, we have

$$\delta f = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3$$

We have already shown that

$$\underline{dr} = h_1 du_1 \underline{e}_1 + h_2 du_2 \underline{e}_2 + h_3 du_3 \underline{e}_3.$$

Using the orthogonality of the basis vectors,  $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ , we can write

$$\begin{aligned} \delta f &= \left( \frac{\partial f}{\partial u_1} \underline{e}_1 + \frac{\partial f}{\partial u_2} \underline{e}_2 + \frac{\partial f}{\partial u_3} \underline{e}_3 \right) \cdot (\underline{e}_1 du_1 + \underline{e}_2 du_2 + \underline{e}_3 du_3) \\ &= \left( \frac{1}{h_1} \frac{\partial f}{\partial u_1} \underline{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \underline{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \underline{e}_3 \right) \cdot (h_1 \underline{e}_1 du_1 + h_2 \underline{e}_2 du_2 + h_3 \underline{e}_3 du_3) \\ &= \left( \frac{1}{h_1} \frac{\partial f}{\partial u_1} \underline{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \underline{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \underline{e}_3 \right) \cdot d\underline{r} \end{aligned}$$

Comparing this result with equation (1) above, we obtain the following expression for  $\underline{\nabla} f$  in orthogonal curvilinears

$$\begin{aligned} \underline{\nabla} f &= \frac{1}{h_1} \frac{\partial f}{\partial u_1} \underline{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \underline{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \underline{e}_3 \\ &= \sum_{i=1}^3 \frac{1}{h_i} \frac{\partial f}{\partial u_i} \underline{e}_i \end{aligned}$$

For **spherical polars**, we obtain

$$\underline{\nabla} f(r, \theta, \phi) = \underline{e}_r \frac{\partial f}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \underline{e}_\phi \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}.$$

For **cylindrical polars**, we obtain

$$\underline{\nabla} f(\rho, \phi, z) = \underline{e}_\rho \frac{\partial f}{\partial \rho} + \underline{e}_\phi \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \underline{e}_z \frac{\partial f}{\partial z}.$$

## Divergence

In orthogonal curvilinear coordinates

$$\operatorname{div} \underline{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right\}$$

This expression can be obtained by using the integral definition of  $\operatorname{div} \underline{A}$ , or alternatively using vector operator identities (see BK 4.13, RHB 8.10).

For Cartesian coordinates, we have  $h_i = 1$ , and we regain the usual expression for  $\underline{\nabla} \cdot \underline{A}$  in Cartesians.

For **spherical polars** we have

$$\begin{aligned} \operatorname{div} \underline{A}(r, \theta, \phi) &= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (r^2 \sin \theta A_r) + \frac{\partial}{\partial \theta} (r \sin \theta A_\theta) + \frac{\partial}{\partial \phi} (r A_\phi) \right\} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \left\{ \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{\partial}{\partial \phi} (A_\phi) \right\}. \end{aligned}$$

where  $A_r$ ,  $A_\theta$ , and  $A_\phi$  are the components of the vector field  $\underline{A}$  in the basis  $(\underline{e}_r, \underline{e}_\theta, \underline{e}_\phi)$ .

For **cylindrical polars** we have

$$\begin{aligned}\operatorname{div}\underline{A}(\rho, \phi, z) &= \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} (\rho A_z) \right\} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z.\end{aligned}$$

where  $A_\rho$ ,  $A_\phi$ , and  $A_z$  are the components of the vector field  $\underline{A}$  in the basis  $(\underline{e}_\rho, \underline{e}_\phi, \underline{e}_z)$ .

### Curl

In orthogonal curvilinear co-ordinates, curl is most conveniently written as

$$\underline{\nabla} \times \underline{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \underline{e}_1 & h_2 \underline{e}_2 & h_3 \underline{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

*e.g.* the first component is given by

$$\underline{e}_1 \cdot \underline{\nabla} \times \underline{A} = \frac{1}{h_2 h_3} \left\{ \frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right\}$$

and the components of  $\underline{\nabla} \times \underline{A}$  in the  $\underline{e}_2$  and  $\underline{e}_3$  directions may be obtained by cyclic permutations of the suffices.

The above formula can be demonstrated by using the line integral definition of curl, as used to prove Stokes' theorem, (see tutorial) or by vector operator identities (BK. 4.13 or RHB 8.10).

For **spherical polars** we have

$$\underline{\nabla} \times \underline{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \underline{e}_r & r \underline{e}_\theta & r \sin \theta \underline{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix}$$

### Laplacian of a Scalar Field

The Laplacian operator acting on a *scalar* field is defined by  $\nabla^2 f = \underline{\nabla} \cdot (\underline{\nabla} f)$ , giving:-

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right\}$$

In **spherical polars**, we have

$$\begin{aligned}\nabla^2 f(r, \theta, \phi) &= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right) \right\} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \left\{ \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial^2 f}{\partial \phi^2} \right\}.\end{aligned}$$