CIHAN UNIVERSITY SULAIMANIYA

Mathematics-1-

For Engineering

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Differentiation

Contents

3.1 Tangents and the Derivative at a Point

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

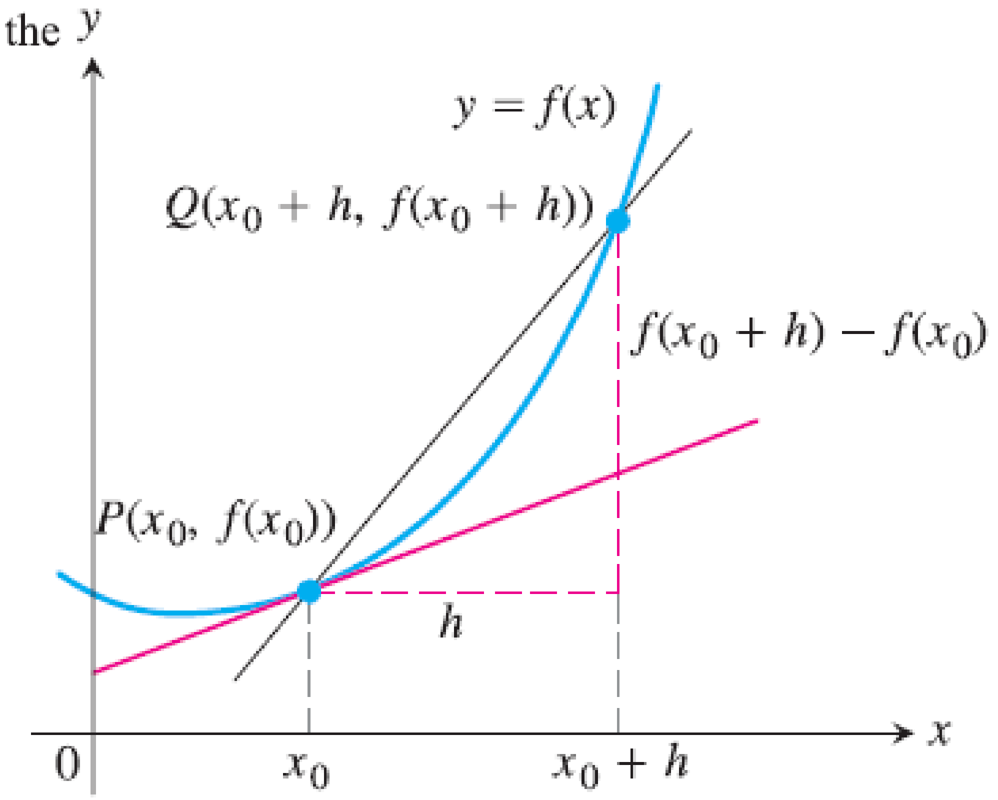


FIGURE The slope of the tangent line at P is $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

3.2 The Derivative as a Function

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

EXAMPLES

1. If $f(x) = x^2$ then

Solution

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} 2x + h$$

$$= 2x.$$

EXAMPLE Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.$$

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

(a) We use the alternative formula to calculate f':

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})}$$

$$= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4

$y = 2 + \frac{1}{4}(x - 4)$ then $y = \frac{1}{4}x + 1.$

Derivative of the Square Root Function

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

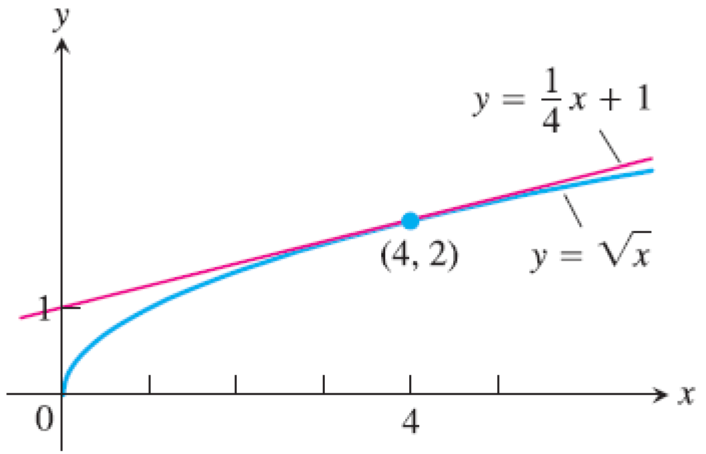


FIGURE The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

exist at the endpoints .

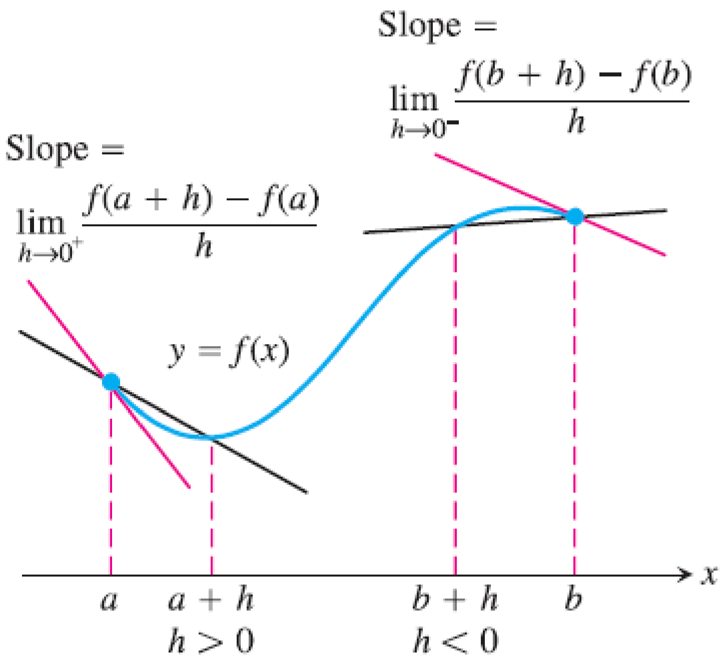


FIGURE Derivatives at endpoints are one-sided limits.

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution the derivative of $y = mx + b$ is the slope m. Thus, to the right of the origin,

To the left, $$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1.$$

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1$$

There is no derivative at the origin because the one-sided derivatives differ there: Right-hand derivative of $|x|$ at zero

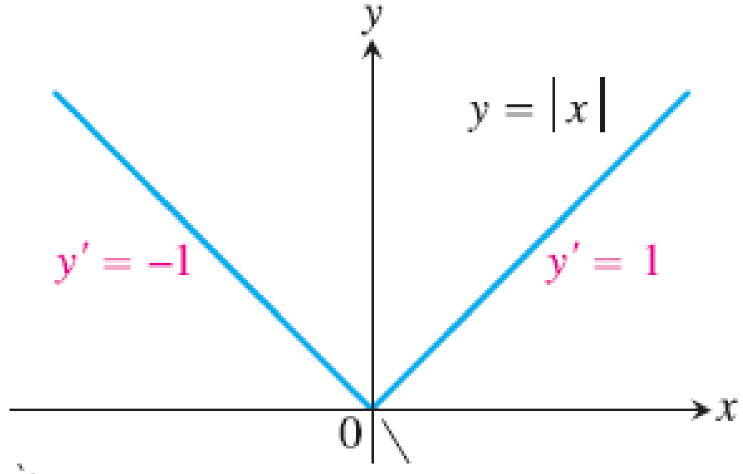
$$= \lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

$|h| = h$ when $h > 0$

Left-hand derivative of $|x|$ at zero $$= \lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h}$$

$$= \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

$|h| = -h$ when $h < 0$



y' not defined at $x = 0$: right-hand derivative $\neq$ left-hand derivative

FIGURE The function $y = |x|$ is not differentiable at the origin where the graph has a "corner"

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

$\lim_{x\to c} f(x) = f(c)$

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$
2. $F(x) = (x-1)^2 + 1; \quad F'(-1), F'(0), F'(2)$
3. $g(t) = \dfrac{1}{t^2}; \quad g'(-1), g'(2), g'(\sqrt{3})$
4. $k(z) = \dfrac{1-z}{2z}; \quad k'(-1), k'(1), k'(\sqrt{2})$
5. $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s+1}; \quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\dfrac{dy}{dx}$ if $y = 2x^3$
8. $\dfrac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$
9. $\dfrac{ds}{dt}$ if $s = \dfrac{t}{2t+1}$
10. $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$
11. $\dfrac{dp}{dq}$ if $p = \dfrac{1}{\sqrt{q+1}}$
12. $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{3w-2}}$

3.3 Differentiation Rules

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Derivative of a Constant Function

If f has the constant value $f(x) = c$, then $\frac{df}{dx} = \frac{d}{dx}(c) = 0.$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE Differentiate the following powers of x.

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

How to Read the Symbols for Derivatives

y'	"y prime"
y''	"y double prime"
$\dfrac{d^2y}{dx^2}$	"d squared y dx squared"
y'''	"y triple prime"
$y^{(n)}$	"y super n"
$\dfrac{d^ny}{dx^n}$	"d to the n of y by dx to the n"
D^n	"D to the n"

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

Exercises 3.3

Derivative Calculations

Find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$
4. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
5. $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$
6. $y = (x-1)(x^2 + 3x - 5)$
7. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$
8. $s = -2t^{-1} + \dfrac{4}{t^2}$
9. $y = (3 - x^2)(x^3 - x + 1)$
10. $y = 4 - 2x - x^{-3}$
11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$
12. $p = \left(\dfrac{q^2 + 3}{12q}\right)\left(\dfrac{q^4 - 1}{q^3}\right)$

Derivatives of Trigonometric Functions

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0, \quad \lim_{h \to 0} \frac{\sin h}{h} = 1$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right)$$

$$= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

EXAMPLE We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$: **(b)** $y = x^2 \sin x$: **(c)** $y = \dfrac{\sin x}{x}$:

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}$$ Derivative definition

$$= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$ Cosine angle sum identity

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 \qquad = -\sin x.$$

EXAMPLE We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5x + \cos x$: **(b)** $y = \sin x \cos x$: **(c)** $y = \dfrac{\cos x}{1 - \sin x}$:

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE Find a. $d(\tan x)/dx$. b. y'' if $y = \sec x$.

Derivatives

In Exercises 1–18, find dy/dx.

1. $y = -10x + 3\cos x$
3. $y = x^2 \cos x$
2. $y = \frac{3}{x} + 5\sin x$
4. $y = \sqrt{x}\sec x + 3$
5. $y = \csc x - 4\sqrt{x} + 7$
6. $y = x^2 \cot x - \frac{1}{x^2}$
7. $f(x) = \sin x \tan x$
8. $g(x) = \csc x \cot x$
9. $y = (\sec x + \tan x)(\sec x - \tan x)$
10. $y = (\sin x + \cos x)\sec x$

11. $y = \frac{\cot x}{1 + \cot x}$
13. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
12. $y = \frac{\cos x}{1 + \sin x}$
14. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$
15. $y = x^2 \sin x + 2x\cos x - 2\sin x$
16. $y = x^2 \cos x - 2x\sin x - 2\cos x$
17. $f(x) = x^3 \sin x \cos x$
18. $g(x) = (2 - x)\tan^2 x$

CHAIN RULE

$$\frac{d}{dx}[f(g(x))] = \frac{df}{dg}\frac{dg}{dx} = f'(g(x))\,g'(x)$$

Differentiate the outer function first then multiply by the derivative of the inner function.

EXAMPLES

1. Since $\frac{d}{dx}\sin x = \cos x$ then

$$\frac{d}{dx}[\sin(x^2)] = \cos(x^2)\,\frac{d}{dx}[x^2]$$
$$= \cos(x^2)2x.$$

2. Since $\frac{d}{dx}x^3 = 3x^2$ and $\frac{d}{dx}\sin x = \cos x$ then

$$\frac{d}{dx}[\sin^3 x] = 3\sin^2 x\ \cos x.$$

The Chain RuLe

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$

2. $y = 2u^3, \quad u = 8x - 1$

3. $y = \sin u, \quad u = 3x + 1$

4. $y = \cos u, \quad u = -x/3$

5. $y = \cos u, \quad u = \sin x$

6. $y = \sin u, \quad u = x - \cos x$

7. $y = \tan u, \quad u = 10x - 5$

8. $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$

10. $y = (4 - 3x)^9$

11. $y = \left(1 - \frac{x}{7}\right)^{-7}$

12. $y = \left(\frac{x}{2} - 1\right)^{-10}$

13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

14. $y = \sqrt{3x^2 - 4x + 6}$

15. $y = \sec(\tan x)$

16. $y = \cot\left(\pi - \frac{1}{x}\right)$

17. $y = \sin^3 x$

18. $y = 5\cos^{-4} x$

IMPLICIT DIFFERENTIATION

To find $y'(x)$ where $y(x)$ is given implicitly, differentiate normally but treat each y as an unknown function of x. For example, if given

$$f(y) = g(x)$$

then differentiating gives

$$f'(y)\frac{dy}{dx} = g'(x) \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{g'(x)}{f'(y)}$$

where the chain rule has been used to obtain the left hand side.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Find dy/dx if $y^2 = x^2 + \sin xy$

PARAMETRIC DIFFERENTIATION

Given $y = f(t)$ and $x = g(t)$, dy/dx may be calculated as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}.$$

EXAMPLES

1. If $y(t) = t^2$ and $x(t) = \sin t$ then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{\cos t}$.
2. If $y(t) = \sin t$ and $x(t) = \cos t$ then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t}{-\sin t} = -\cot t$.

Derivatives of Inverses of Differentiable Functions

Example:

If *f(x)* = (1/2)x +1 , find the derivative of inverse of f(x)

Solution:the inverse is

f (x) = 2x – 2

Then d*f*/dx is 2

The Derivative of y = ln x. For every positive value of x, we have $$\frac{d}{dx}\ln x = \frac{1}{x},$$

and the Chain Rule extenda this formula for positive functions *u(x)*:

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0.$$

EXAMPLE 1 find derivatives.

(a) $$\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$$

(b) $$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3)$$

$$= \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

EXAMPLE 5 Find dy/dx if $$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

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Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
&= \ln ((x^2+1)(x+3)^{1/2}) - \ln (x-1) \\
&= \ln (x^2+1) + \ln (x+3)^{1/2} - \ln (x-1) \\
&= \ln (x^2+1) + \frac{1}{2}\ln (x+3) - \ln (x-1).
\end{aligned}$$

We then take derivatives of both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx: $$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y from the original equation:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Find the 1st and 2nd derivative following problems

$y = \ln (t^2)$ $\qquad y = \ln \frac{3}{x}$ $\qquad y = \ln (2\theta + 2)$

$y = t(\ln t)^2$ $\qquad y = (\ln x)^3$ $\qquad y = \frac{\ln x}{1 + \ln x}$

$y = \frac{x \ln x}{1 + \ln x}$ $\qquad y = \ln (\sec \theta + \tan \theta)$

$y = \ln (\ln (\ln x))$ $\qquad y = \theta(\sin (\ln \theta) + \cos (\ln \theta))$

$y = \ln \left(\frac{\sqrt{\sin \theta \cos \theta}}{1 + 2 \ln \theta}\right)$ $\qquad y = \ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

The Derivative of e^x

That is, for $y = e^x$, we find that $dy/dx = e^x$

The Chain Rule extends the derivative result for the natural exponential function to a more general form involving a function $u(x)$:

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

Below a few examples of derivative

(a) $\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x = 5e^x$

(b) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (2) with $u = -x$

(c) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cdot\cos x$ Eq. (2) with $u = \sin x$

(d) $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}}\cdot\frac{d}{dx}\left(\sqrt{3x+1}\right)$ Eq. (2) with $u = \sqrt{3x+1}$

$$= e^{\sqrt{3x+1}}\cdot\frac{1}{2}(3x+1)^{-1/2}\cdot 3 = \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$$

The General Exponential Function a^u

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DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x \ln a}.$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a)$$
$$= a^x \ln a.$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}.$$

EXAMPLE 5

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \frac{d}{dx}(-x) = -3^{-x} \ln 3$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3) \frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$

the 2nd derivative is

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

EXAMPLE Differentiate $f(x) = x^x$, $x > 0$.

Solution We cannot apply the power rule here because the exponent is the *variable* x rather than being a constant value n (rational or irrational). However, from the definition of the general exponential function we note that $f(x) = x^x = e^{x \ln x}$, and differentiation gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x \ln x}) \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) && \text{Eq. (2) with } u = x \ln x \\ &= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) \\ &= x^x(\ln x + 1). && x > 0 \end{aligned}$$

■

Derivatives and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$$

note

$$\log_a x = \frac{\ln x}{\ln a}.$$

EXAMPLE

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

Find the derivative of the following

$y = x^{\pi}$

$y = (\cos\theta)^{\sqrt{2}}$

$y = 7^{\sec\theta}\ln 7$

$y = 2^{\sin 3t}$

$y = \log_2 5\theta$

$y = \log_4 x + \log_4 x^2$

$y = x^3 \log_{10} x$

$$y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$$

$y = t^{1-e}$

$y = (\ln\theta)^{\pi}$

$y = 3^{\tan\theta}\ln 3$

$y = 5^{-\cos 2t}$

$y = \log_3(1 + \theta\ln 3)$

$y = \log_{25} e^x - \log_5\sqrt{x}$

$y = \log_3 r \cdot \log_9 r$

$$y = \log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right)$$

$$y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right)$$

$y = \theta\sin(\log_7\theta)$

$y = \log_{10} e^x$

$y = 3^{\log_2 t}$

$y = \log_2(8t^{\ln 2})$

$y = 3\log_8(\log_2 t)$

SECOND DERIVATIVE

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The second (or double) derivative is the derivative of the derivative:

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right).$$

Higher derivatives are found by repeated differentiation.

EXAMPLES

1. If $f(x) = x^4$ then $f'(x) = 4x^3$ and $f''(x) = 12x^2$.

2. If $s(t) = e^{2t}$ is the position of a particle with time t, then $s'(t) = 2e^{2t}$ is the velocity and $s''(t) = 4e^{2t}$ is the acceleration.

STATIONARY POINTS

A **stationary point** is a point (x, y) where $f'(x) = 0$. At this point the tangent to the graph is flat.

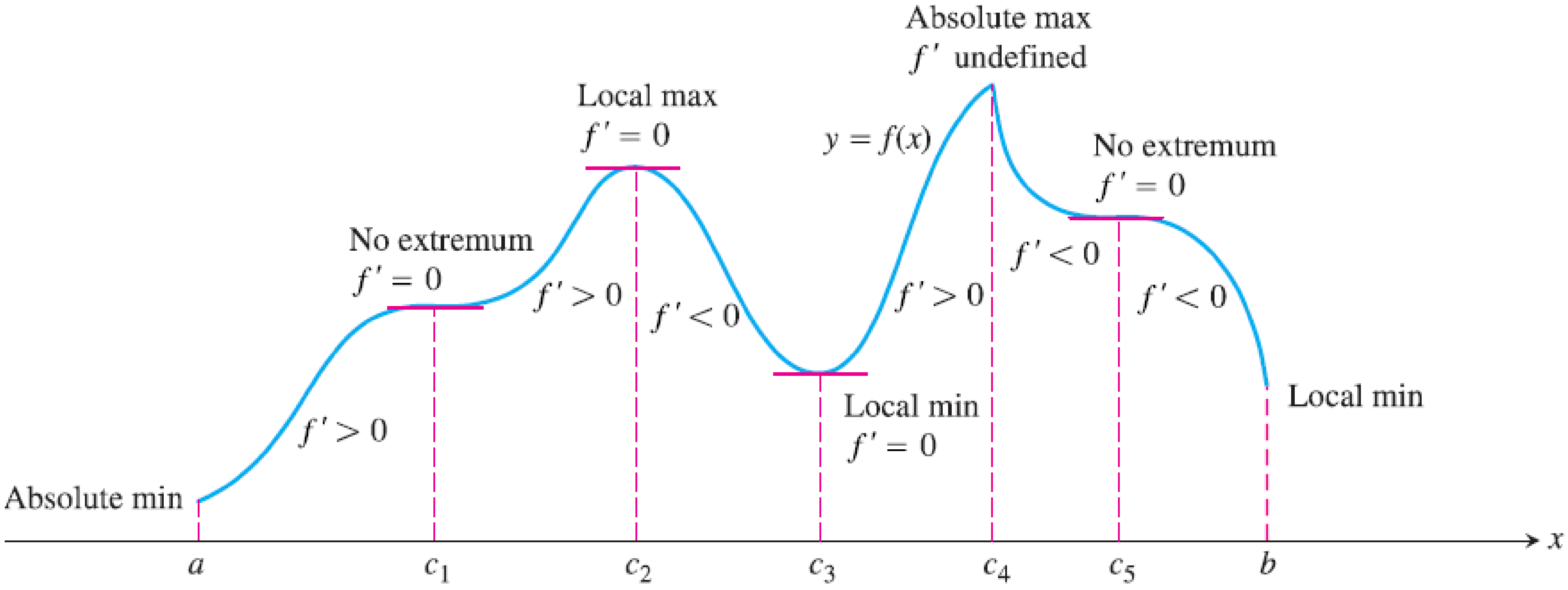


FIGURE The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

EXAMPLES

1. The function $y = x^2 + 2x + 2$ has a stationary point when

$$\frac{dy}{dx} = 2x + 2 = 0 \implies x = -1.$$

2. The function $y = 2x^3 - 9x^2 + 12x$ has stationary points when

$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 0 \implies x = 1, 2.$$

3. The function $y = xe^{-x}$ has a maximum when

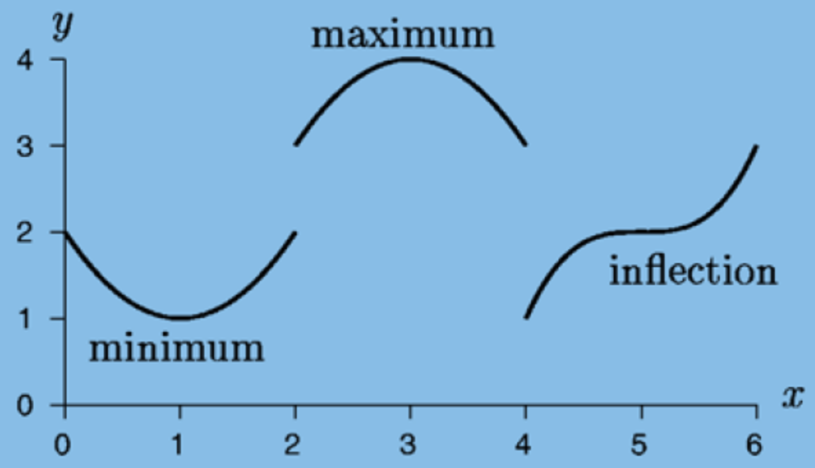
$$\frac{dy}{dx} = e^{-x}(x - 1) = 0 \implies x = 1.$$

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

A **local maximum** is when the function at the stationary point is higher than the surrounding points. A **local minimum** is lower than the surrounding points. An **inflection** point is where the graph is flat but neither a maximum nor minimum.



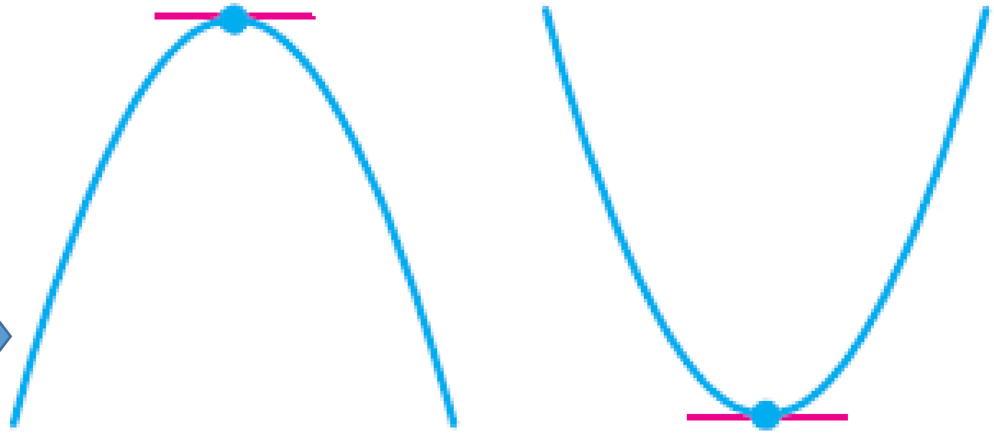
The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

At a stationary point $x = a$ the **second derivative** indicates the type of stationary point:

1. if $f''(a) > 0$ then $x = a$ is a local minimum
2. if $f''(a) < 0$ then $x = a$ is a local maximum.
3. if $f''(a) = 0$ then $x = a$ is an inflection point.



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

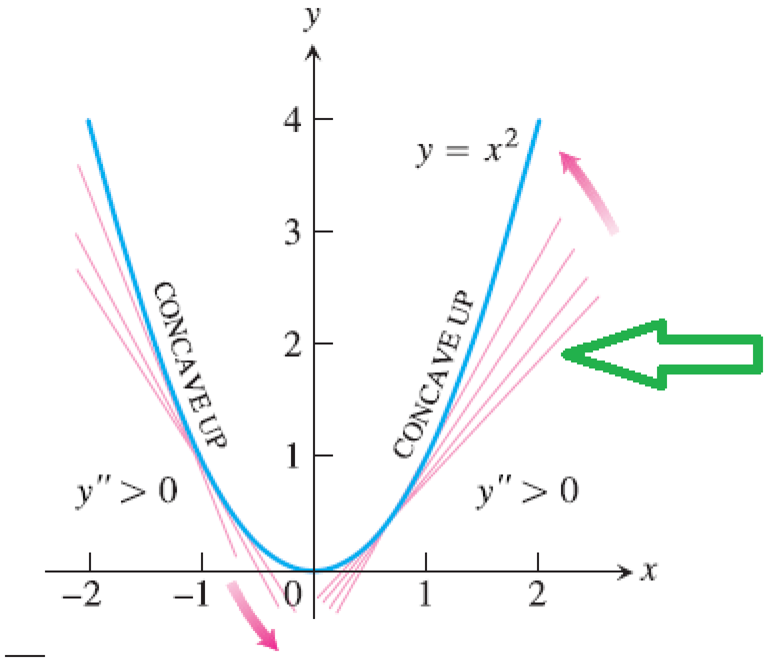


FIGURE The graph of $f(x) = x^2$ is concave up on every interval

EXAMPLE 1

(a) The curve $y = x^3$ (Figure 4.23) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.24) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.25). ■

DEFINITION A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

EXAMPLE 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

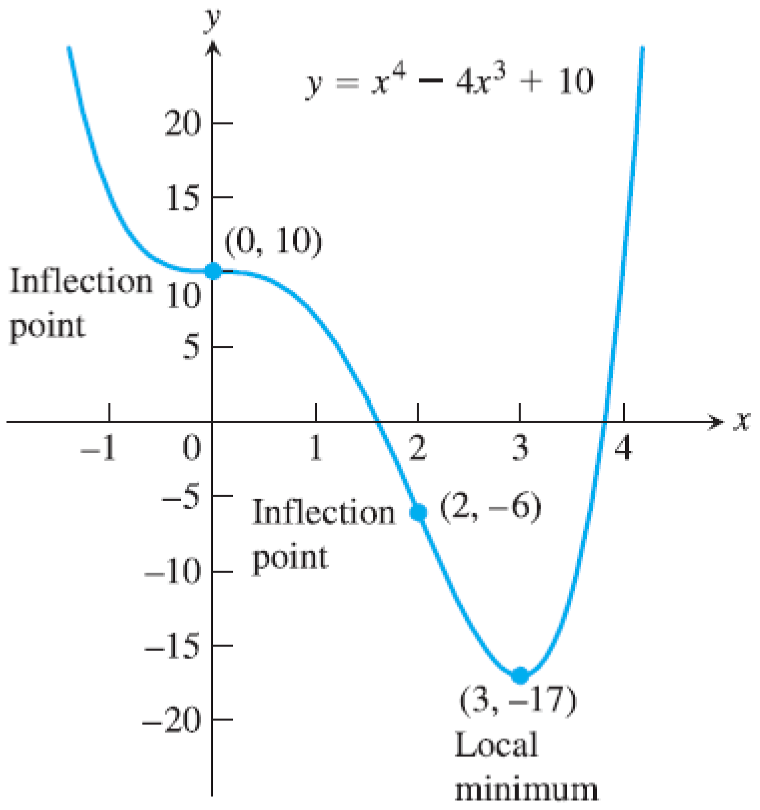
(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	+	−	+
Behavior of f	concave up	concave down	concave up

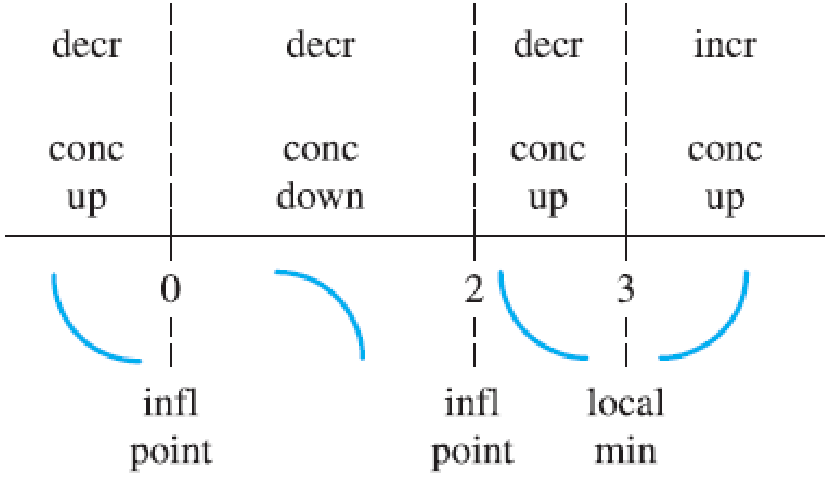
We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

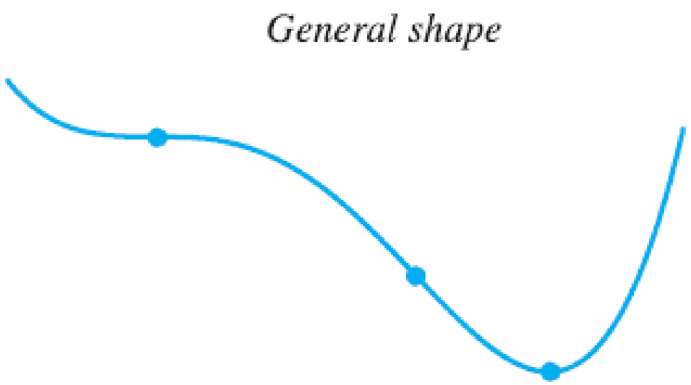


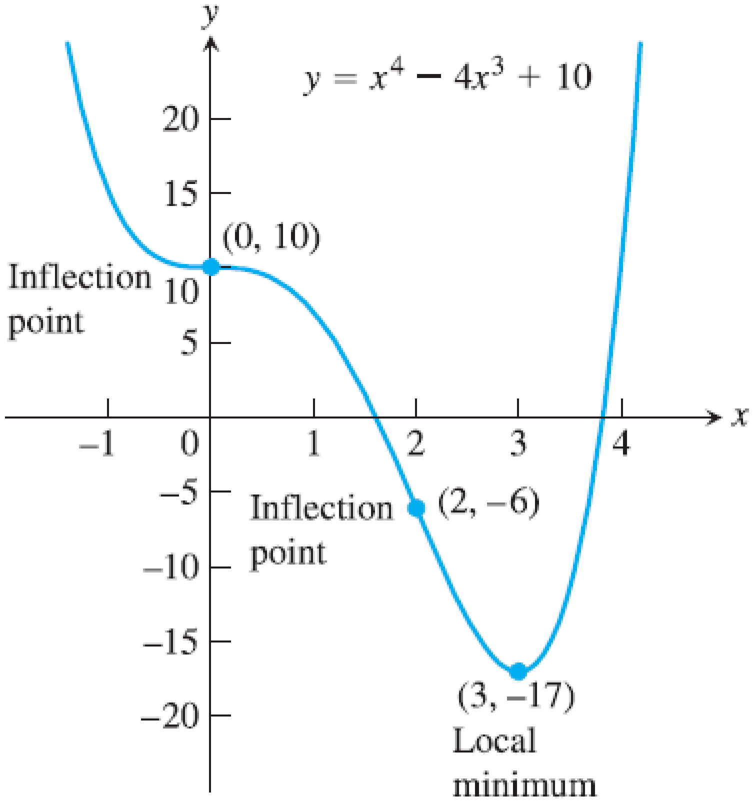
(d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

The general shape of the curve is shown in the accompanying figure.







THANKS FOR YOUR ATTENTION