CIHAN UNIVERSITY
SULAIMANIYA

Mathematics-1-

For Engineering

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Chapter 5
Integration

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Contents

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Integration

INTEGRATION

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The *integral* is of fundamental importance in statistics, the sciences, and engineering. We use it to calculate quantities ranging from probabilities and averages to energy consumption and the forces against a
dam's floodgates.

The idea behind the integral is that we can effectively compute such quantities by breaking them into small pieces and then summing the contributions from each piece

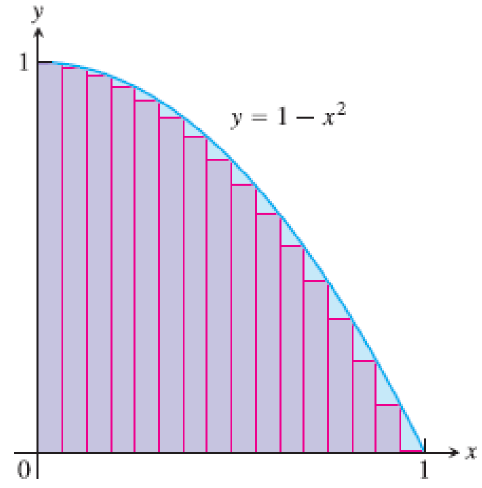
The basis for formulating definite integrals is the construction of appropriate finite sums.

The area under the graph of a positive function

The area under the graph of a positive function,

If the interval $[a, b]$ is subdivided into n subintervals of equal widths $\Delta x = (b - a)/n$, and if $f(c_k)$ is the value of f at the chosen point c_k in the kth subinterval, this process gives a finite sum of the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x$$



(a)

EXAMPLE Estimate the average value of the function $j(x) = \sin x$ on the interval

$$A \approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right)\cdot\frac{\pi}{8}$$

$$\approx (.38 + .71 + .92 + 1 + 1 + .92 + .71 + .38)\cdot\frac{\pi}{8} = (6.02)\cdot\frac{\pi}{8} \approx 2.365.$$

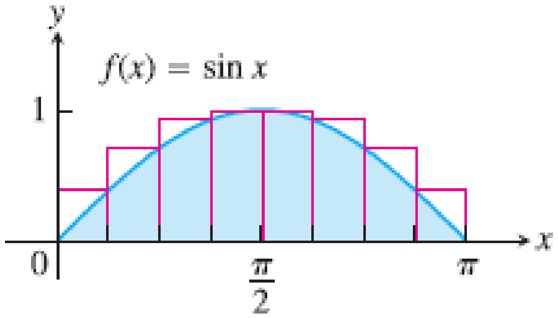


FIGURE 5.7 Approximating the area under $f(x) = \sin x$ between 0 and π to compute the average value of $\sin x$ over $[0, \pi]$, using eight rectangles (Example 4).

The Definite Integral

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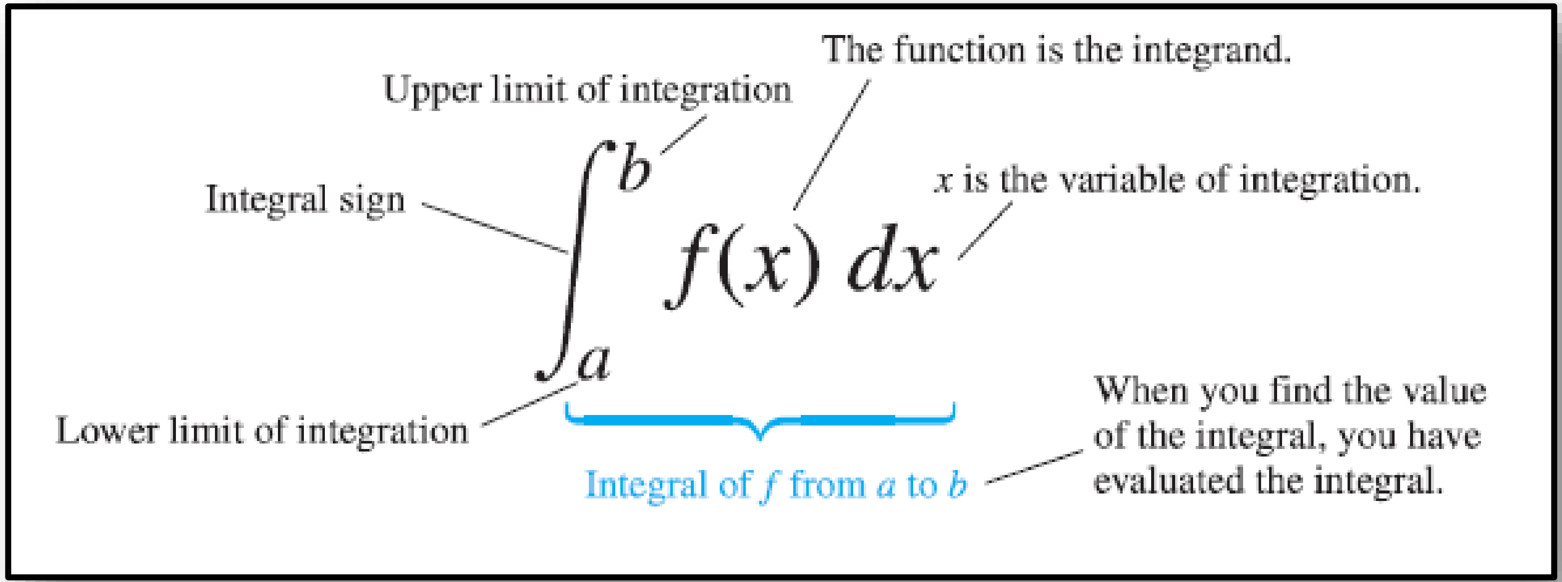
DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon .$$



THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

Properties of Definite Integrals

In defining $\int_a^b f(x)\,dx$ as a limit of sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$,

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Since $a = b$ gives $\Delta x = 0$, whenever $f(a)$ exists we define

$$\int_a^a f(x)\,dx = 0.$$

TABLE Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

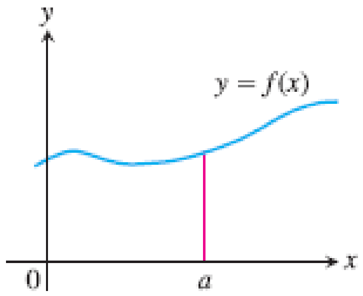
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

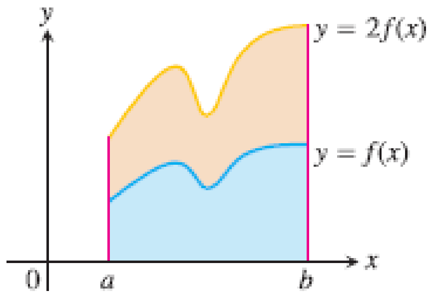
7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)



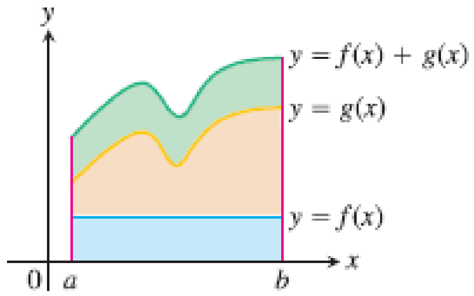
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



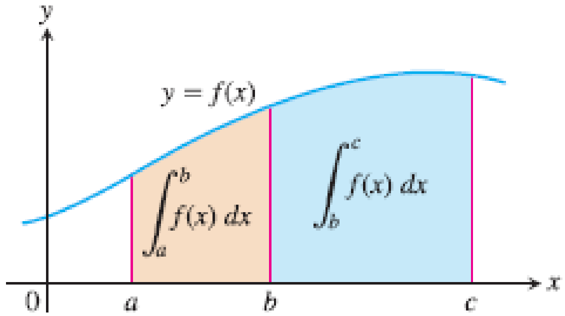
(b) *Constant Multiple:* (*k* = 2)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



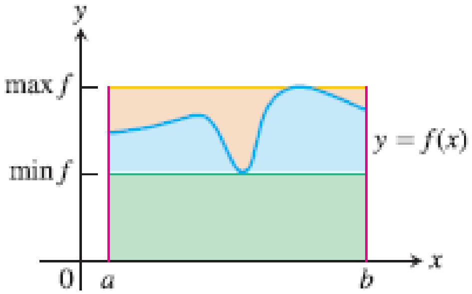
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



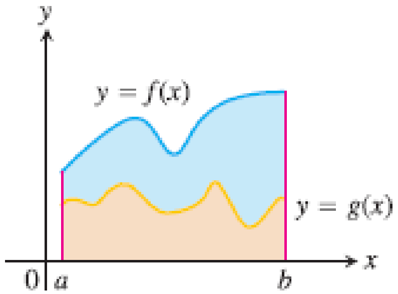
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$



(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Indefinite Integrals

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The **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f, symbolized by

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

Substitution: Running the Chain Rule Backwards

EXAMPLE 1 Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$$

so that by substitution we have

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du = \frac{u^6}{6} + C = \frac{(x^3 + x)^6}{6} + C$$

Substitute $x^3 + x$ for u.

H.W

Find $\int \sqrt{2x+1}\,dx.$

Evaluate $\int x\sqrt{2x+1}\,dx.$

Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}.$

Evaluating Indefinite Integrals

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1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$

Exercises 5

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1. $\int 2(2x+4)^5\,dx, \quad u = 2x+4$

2. $\int 7\sqrt{7x-1}\,dx, \quad u = 7x-1$

3. $\int 2x(x^2+5)^{-4}\,dx, \quad u = x^2+5$

4. $\int \frac{4x^3}{(x^4+1)^2}\,dx, \quad u = x^4+1$

5. $\int (3x+2)(3x^2+4x)^4\,dx, \quad u = 3x^2+4x$

6. $\int \frac{\left(1+\sqrt{x}\right)^{1/3}}{\sqrt{x}}\,dx, \quad u = 1+\sqrt{x}$

7. $\int \sin 3x\,dx, \quad u = 3x$

8. $\int x\sin(2x^2)\,dx, \quad u = 2x^2$

9. $\int \sec 2t \tan 2t\,dt, \quad u = 2t$

10. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1-\cos\frac{t}{2}$

11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1-r^3$

12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4+4y^2+1$

13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2}-1$

14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

15. $\int \csc^2 2\theta \cot 2\theta\,d\theta$

    a. Using $u = \cot 2\theta$

    b. Using $u = \csc 2\theta$

12. $\int \tan x \, dx = \ln |\sec x| + C$

13. $\int \cot x \, dx = \ln |\sin x| + C$

14. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

15. $\int \csc x \, dx = -\ln |\csc x + \cot x| + C$

16. $\int \sinh x \, dx = \cosh x + C$

17. $\int \cosh x \, dx = \sinh x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x >$

8. $\int \sec^2 x \, dx = \tan x + C$

9. $\int \csc^2 x \, dx = -\cot x + C$

10. $\int \sec x \tan x \, dx = \sec x + C$

11. $\int \csc x \cot x \, dx = -\csc x + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x >$

$$\int \sin^m x \cos^n x \, dx,$$

Case 1 If ***m* is odd**, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both** m **and** n **are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the integrand to one in lower powers of cos 2*x*.

EXAMPLE Evaluate $\int \sin^3 x \cos^2 x\, dx$.

Solution

$$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x \sin x\, dx \qquad m \text{ is odd.}$$

$$= \int (1 - \cos^2 x) \cos^2 x\, (-d(\cos x)) \qquad \sin x\, dx = -d(\cos x)$$

$$= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x$$

$$= \int (u^4 - u^2)\, du \qquad \text{Multiply terms.}$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

EXAMPLE 2 Evaluate $\int \cos^5 x\,dx$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx = \int (1 - \sin^2 x)^2\,d(\sin x) \qquad \cos x\,dx = d(\sin x)$$

$$= \int (1 - u^2)^2\,du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\,du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

■

EXAMPLE 3 Evaluate $\int \sin^2 x \cos^4 x \, dx$.

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

m and *n* both even

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x)\cos 2x \, dx$$

$u = \sin 2x,$
$du = 2\cos 2x \, dx$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

EXAMPLE Evaluate $\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \quad \text{or} \quad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes ⇨ $1 + \cos 4x = 2\cos^2 2x.$ Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$$

$\cos 2x \geq 0$ on $[0, \pi/4]$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.$$

■

The Integrals of $\sin^2 x$ and $\cos^2 x$

EXAMPLE

(a) $$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx$$

$$= \frac{1}{2}\int (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2}\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Integrals of Powers of tan x and sec x

To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE Evaluate $\int \tan^4 x\, dx$.

Solution

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$
$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$
$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$
$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx.$$

In the first integral, we let $u = \tan x, \quad du = \sec^2 x\, dx$

and have $\int u^2\, du = \frac{1}{3}u^3 + C_1$. The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \quad \int \sin mx \cos nx\, dx, \quad \text{and} \quad \int \cos mx \cos nx\, dx$$

We can evaluate these integrals through integration by parts, It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \qquad (3)$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \qquad (4)$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \qquad (5)$$

EXAMPLE Evaluate $\int \sin 3x \cos 5x\, dx$.

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\int \sin 3x \cos 5x\, dx = \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx = \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

Exercises

Powers of Sines and Cosines

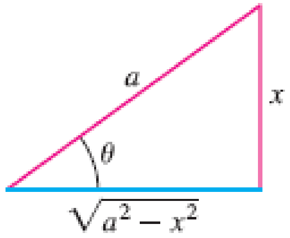
Evaluate the integrals in Exercises

1. $\int \cos 2x\, dx$
2. $\int_0^{\pi} 3 \sin \frac{x}{3}\, dx$
3. $\int \cos^3 x \sin x\, dx$
4. $\int \sin^4 2x \cos 2x\, dx$
5. $\int \sin^3 x\, dx$
6. $\int \cos^3 4x\, dx$
7. $\int \sin^5 x\, dx$
8. $\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$
9. $\int \cos^3 x\, dx$
10. $\int_0^{\pi/6} 3 \cos^5 3x\, dx$
11. $\int \sin^3 x \cos^3 x\, dx$
12. $\int \cos^3 2x \sin^5 2x\, dx$
13. $\int \cos^2 x\, dx$
14. $\int_0^{\pi/2} \sin^2 x\, dx$
15. $\int_0^{\pi/2} \sin^7 y\, dy$
16. $\int 7 \cos^7 t\, dt$

27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}}\, dx$
28. $\int_0^{\pi/6} \sqrt{1 + \sin x}\, dx$

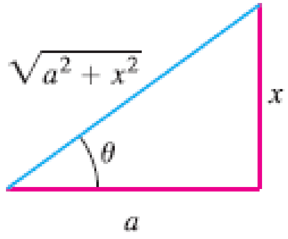
    $\left(\textit{Hint}: \text{Multiply by } \sqrt{\frac{1 - \sin x}{1 - \sin x}}.\right)$
29. $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1 - \sin x}}\, dx$
30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x}\, dx$
31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta}\, d\theta$
32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2}\, dt$
33. $\int \sec^2 x \tan x\, dx$
34. $\int \sec x \tan^2 x\, dx$
35. $\int \sec^3 x \tan x\, dx$
36. $\int \sec^3 x \tan^3 x\, dx$
37. $\int \sec^2 x \tan^2 x\, dx$
38. $\int \sec^4 x \tan^2 x\, dx$

Trigonometric Substitutions



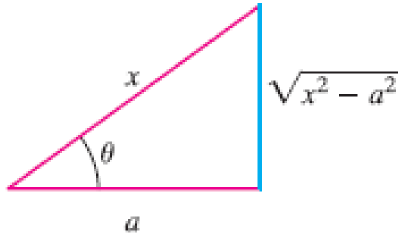
$x = a \sin \theta$

$\sqrt{a^2 - x^2} = a|\cos \theta|$



$x = a \tan \theta$

$\sqrt{a^2 + x^2} = a|\sec \theta|$



$x = a \sec \theta$

$\sqrt{x^2 - a^2} = a|\tan \theta|$

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

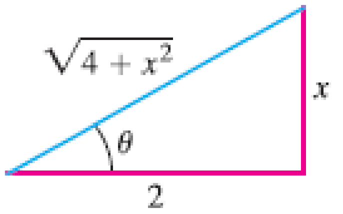
FIGURE Reference triangles for the tree basic substitutions identifying the sides labeled x and a for eah substitution.

EXAMPLE Evaluate $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta.$$



$x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2}$

and $\sec\theta = \frac{\sqrt{4+x^2}}{2}$.

Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C. \qquad \text{From Fig. 8.4}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$

EXAMPLE 2 Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

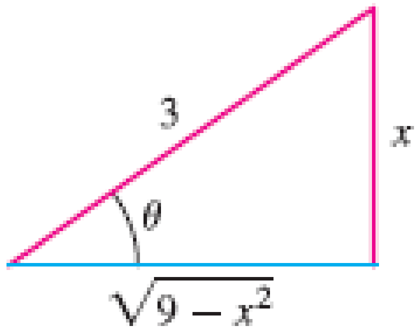
$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C \qquad \text{Fig. 8.5}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$



$x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

Using Trigonometric Substitutions

Evaluate the integrals in Exercises

1. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$

2. $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

10. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

11. $\displaystyle\int \frac{100}{36+25x^2}\,dx$

12. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

13. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

14. $\displaystyle\int \frac{x\,dx}{25+4x^2}$

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$
$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x-3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

EXAMPLE Use partial fractions to evaluate

1; $$\int \frac{6x + 7}{(x + 2)^2}\, dx. \qquad = 6 \ln |x + 2| + 5(x + 2)^{-1} + C.$$

2, $$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx = x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C.$$

EXAMPLE Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\, dx.$$

$$= \ln (x^2 + 1) + \tan^{-1} x - 2 \ln |x - 1| - \frac{1}{x - 1} + C.$$

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1. $\int (\sin x + e^x)\, dx = -\cos x + e^x + c$

2. $\int 5\cos x\, dx = 5\sin x + c$

17. $\int \sqrt{3 - 2s}\, ds$

18. $\int \frac{1}{\sqrt{5s + 4}}\, ds$

29. $\int x^{1/2} \sin(x^{3/2} + 1)\, dx$

19. $\int \theta \sqrt[4]{1 - \theta^2}\, d\theta$

20. $\int 3y\sqrt{7 - 3y^2}\, dy$

30. $\int \csc\left(\frac{v - \pi}{2}\right) \cot\left(\frac{v - \pi}{2}\right) dv$

21. $\int \frac{1}{\sqrt{x}\,(1 + \sqrt{x})^2}\, dx$

22. $\int \cos(3z + 4)\, dz$

31. $\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)}\, dt$

32. $\int \frac{\sec z \tan z}{\sqrt{\sec z}}\, dz$

23. $\int \sec^2(3x + 2)\, dx$

24. $\int \tan^2 x \sec^2 x\, dx$

33. $\int \frac{1}{t^2} \cos\left(\frac{1}{t} - 1\right) dt$

34. $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t} + 3)\, dt$

25. $\int \sin^5 \frac{x}{3} \cos \frac{x}{3}\, dx$

26. $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\, dx$

35. $\int \frac{1}{\theta^2} \sin\frac{1}{\theta} \cos\frac{1}{\theta}\, d\theta$

36. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\, \sin^2\sqrt{\theta}}\, d\theta$

27. $\int r^2 \left(\frac{r^3}{18} - 1\right)^5 dr$

28. $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$

37. $\int t^3 (1 + t^4)^3\, dt$

38. $\int \sqrt{\frac{x - 1}{x^5}}\, dx$

Evaluating Definite Integrals

Evaluating Definite Integrals

Use the Substitution ! to evaluate the integrals in Exercises

1. a. $\int_0^3 \sqrt{y+1}\, dy$ b. $\int_{-1}^0 \sqrt{y+1}\, dy$
2. a. $\int_0^1 r\sqrt{1-r^2}\, dr$ b. $\int_{-1}^1 r\sqrt{1-r^2}\, dr$
3. a. $\int_0^{\pi/4} \tan x \sec^2 x\, dx$ b. $\int_{-\pi/4}^0 \tan x \sec^2 x\, dx$
4. a. $\int_0^{\pi} 3\cos^2 x \sin x\, dx$ b. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx$
5. a. $\int_0^1 t^3(1+t^4)^3\, dt$ b. $\int_{-1}^1 t^3(1+t^4)^3\, dt$
6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\, dt$ b. $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\, dt$
7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\, dr$ b. $\int_0^1 \frac{5r}{(4+r^2)^2}\, dr$
8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$ b. $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$
9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$

EXAMPLE Here we recognize an integral of the form $\int \frac{du}{u}$.

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$u = x^2 - 5,\quad du = 2x\,dx,$
$u(0) = -5,\quad u(2) = -1$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

The Integrals of tan *x*, cot *x*, sec *x*, and csc *x*

Equation (5) tells us how to integrate these trigonometric functions.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$

$u = \cos x > 0$ on $(-\pi/2, \pi/2)$,
$du = -\sin x\,dx$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C.$$

Reciprocal Rule

To integrate $\sec x$, we multiply and divide by $(\sec x + \tan x)$.

$$\int \sec x\, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C$$

$u = \sec x + \tan x$
$du = (\sec x \tan x + \sec^2 x)\, dx$

For $\csc x$, we multiply and divide by $(\csc x + \cot x)$.

$$\int \csc x\, dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

$$= \int \frac{-du}{u} = -\ln |u| + C = -\ln |\csc x + \cot x| + C$$

$u = \csc x + \cot x$
$du = (-\csc x \cot x - \csc^2 x)\, dx$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u \, du = \ln |\sec u| + C \qquad \int \sec u \, du = \ln |\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C \qquad \int \csc u \, du = -\ln |\csc u + \cot u| + C$$

Exponential Functions

If u is any differentiable function of x, then $\dfrac{d}{dx}e^u = e^u \dfrac{du}{dx}.$

The general antiderivative of the exponential function

$$\int e^u\,du = e^u + C$$

Examples

(a) $\displaystyle\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du$ $\quad u = 3x,\ \frac{1}{3}du = dx,\ u(0) = 0,$

$u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du = \frac{1}{3}e^u\Big]_0^{\ln 8} = \frac{1}{3}(8-1) = \frac{7}{3}$$

(b) $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2} = e^1 - e^0 = e - 1$

The General Exponential Function a"

DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x \ln a}.$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}.$$

The general antiderivative $\displaystyle\int a^u \, du = \frac{a^u}{\ln a} + C.$

EXAMPLE

(a) $\displaystyle\int 2^x \, dx = \frac{2^x}{\ln 2} + C$ $a = 2, u = x$

(b) $\displaystyle\int 2^{\sin x} \cos x \, dx = \int 2^u \, du = \frac{2^u}{\ln 2} + C$ $u = \sin x$, $du = \cos x \, dx$,

$$= \frac{2^{\sin x}}{\ln 2} + C$$

u replaced by $\sin x$

(c) $\displaystyle\frac{d}{dx} \log_{10}(3x + 1) = \frac{1}{\ln 10} \cdot \frac{1}{3x + 1} \frac{d}{dx}(3x + 1)$

$$= \frac{3}{(\ln 10)(3x + 1)}$$

(d) $\displaystyle\int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2} \int \frac{\ln x}{x} dx = \frac{1}{\ln 2} \int u \, du$ $\log_2 x = \frac{\ln x}{\ln 2}$

$$= \frac{1}{\ln 2} \frac{u^2}{2} + C = \frac{1}{\ln 2} \frac{(\ln x)^2}{2} + C$$

$$= \frac{(\ln x)^2}{2 \ln 2} + C$$

$u = \ln x$, $du = \frac{1}{x} dx$

Solve the following integrals

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1. $\int_{\ln 4}^{\ln 9} e^{x/2}\,dx$

2. $\int_{0}^{\ln 16} e^{x/4}\,dx$

3. $\int 2t\,e^{-t^2}\,dt$

4. $\int t^3 e^{(t^4)}\,dt$

5. $\int \frac{e^{-1/x^2}}{x^3}\,dx$

6. $\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr$

7. $\int_{\pi/4}^{\pi/2} (1 + e^{\cot\theta})\csc^2\theta\,d\theta$

8. $\int e^{\sec \pi t}\sec \pi t \tan \pi t\,dt$

9. $\int \frac{dx}{1 + e^x}$

10. $\int \frac{e^r}{1 + e^r}\,dr$

11. $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v\,dv$

12. $\int_{0}^{\sqrt{\ln \pi}} 2x\,e^{x^2}\cos(e^{x^2})\,dx$

. $\int 5^x\,dx$

. $\int \frac{3^x}{3 - 3^x}\,dx$

. $\int 3x^{\sqrt{3}}\,dx$

. $\int x^{\sqrt{2}-1}\,dx$

. $\int_{0}^{9} \frac{2\log_{10}(x+1)}{x+1}\,dx$

. $\int_{0}^{1} 2^{-\theta}\,d\theta$

. $\int_{-2}^{0} 5^{-\theta}\,d\theta$

. $\int_{0}^{3} (\sqrt{2}+1)x^{\sqrt{2}}\,dx$

. $\int_{1}^{e} x^{(\ln 2)-1}\,dx$

. $\int \frac{dx}{x\log_{10} x}$

. $\int_{1}^{\sqrt{2}} x2^{(x^2)}\,dx$

. $\int_{1}^{4} \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

. $\int \frac{\log_{10} x}{x}\,dx$

. $\int_{1}^{4} \frac{\log_2 x}{x}\,dx$

. $\int_{0}^{\pi/2} 7^{\cos t}\sin t\,dt$

. $\int_{0}^{\pi/4} \left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$

. $\int_{1}^{4} \frac{\ln 2 \log_2 x}{x}\,dx$

. $\int_{1}^{e} \frac{2\ln 10 \log_{10} x}{x}\,dx$

. $\int_{2}^{3} \frac{2\log_2(x-1)}{x-1}\,dx$

. $\int_{2}^{4} x^{2x}(1 + \ln x)\,dx$

. $\int \frac{x2^{x^2}}{1 + 2^{x^2}}\,dx$

. $\int_{0}^{2} \frac{\log_2(x+2)}{x+2}\,dx$

. $\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}\,dx$

. $\int \frac{dx}{x(\log_8 x)^2}$

TABLE Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

Inverse trigonometric Functions

TABLE Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$ $a = 1, u = x$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$$ $a = \sqrt{3}$, $u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C \quad = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

(c) $$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}}$$ $u = e^x$, $du = e^x\,dx$, $dx = du/e^x = du/u$, $a = \sqrt{6}$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}} \qquad = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

Evaluate the integrals

3. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}$
4. $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}}$
5. $\displaystyle\int \frac{dx}{17 + x^2}$
6. $\displaystyle\int \frac{dx}{9 + 3x^2}$
7. $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 2}}$
8. $\displaystyle\int \frac{dx}{x\sqrt{5x^2 - 4}}$

(a) $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$

Solution we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4$$
$$= 4 - (x - 2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x - 2)^2}}$$
$$= \int \frac{du}{\sqrt{a^2 - u^2}} \qquad a = 2,\ u = x - 2, \text{ and } du = dx$$
$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$= \sin^{-1}\left(\frac{x - 2}{2}\right) + C$$

H.W

(b) $\displaystyle\int \frac{dx}{4x^2 + 4x + 2} = \frac{1}{2}\tan^{-1}(2x + 1) + C$

hint $4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

1. $\displaystyle\int_0^2 \frac{dt}{8 + 2t^2}$
2. $\displaystyle\int_{-2}^2 \frac{dt}{4 + 3t^2}$
3. $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$
4. $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$
5. $\displaystyle\int \frac{3\,dr}{\sqrt{1 - 4(r - 1)^2}}$
6. $\displaystyle\int \frac{6\,dr}{\sqrt{4 - (r + 1)^2}}$
7. $\displaystyle\int \frac{dx}{2 + (x - 1)^2}$
8. $\displaystyle\int \frac{dx}{1 + (3x + 1)^2}$
9. $\displaystyle\int \frac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$

1. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1 + (\sin\theta)^2}$

2. $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1 + (\cot x)^2}$

3. $\displaystyle\int_{0}^{\ln\sqrt{3}} \frac{e^x\, dx}{1 + e^{2x}}$

4. $\displaystyle\int_{1}^{e^{\pi/4}} \frac{4\, dt}{t(1 + \ln^2 t)}$

5. $\displaystyle\int \frac{y\, dy}{\sqrt{1 - y^4}}$

6. $\displaystyle\int \frac{\sec^2 y\, dy}{\sqrt{1 - \tan^2 y}}$

7. $\displaystyle\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

8. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

9. $\displaystyle\int_{-1}^{0} \frac{6\, dt}{\sqrt{3 - 2t - t^2}}$

0. $\displaystyle\int_{1/2}^{1} \frac{6\, dt}{\sqrt{3 + 4t - 4t^2}}$

Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Examples

(a)
$$\int \coth 5x\, dx = \int \frac{\cosh 5x}{\sinh 5x} dx = \frac{1}{5}\int \frac{du}{u} \qquad u = \sinh 5x,\ du = 5\cosh 5x\, dx$$
$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

(b)
$$\int_0^1 \sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx = \frac{1}{2}\int_0^1 (\cosh 2x - 1)\, dx$$
$$= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 = \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$

(c)
$$\int_0^{\ln 2} 4e^x \sinh x\, dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx$$
$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$
$$= 4 - 2\ln 2 - 1 \approx 1.6137$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \qquad u \neq 0$$

Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Evaluating Integrals

1. $\int e^x \sin(e^x)\,dx$
2. $\int e^t \cos(3e^t - 2)\,dt$
3. $\int e^x \sec^2(e^x - 7)\,dx$
4. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\,dy$
5. $\int \sec^2(x) e^{\tan x}\,dx$
6. $\int \csc^2 x\, e^{\cot x}\,dx$
7. $\int_{-1}^{1} \frac{dx}{3x - 4}$
8. $\int_{1}^{e} \frac{\sqrt{\ln x}}{x}\,dx$
9. $\int_{0}^{\pi} \tan\frac{x}{3}\,dx$
0. $\int_{1/6}^{1/4} 2\cot \pi x\,dx$

1. $\int_0^4 \frac{2t}{t^2 - 25}\,dt$
2. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\,dt$
3. $\int \frac{\tan(\ln v)}{v}\,dv$
4. $\int \frac{dv}{v \ln v}$
5. $\int \frac{(\ln x)^{-3}}{x}\,dx$
6. $\int \frac{\ln(x - 5)}{x - 5}\,dx$
7. $\int \frac{1}{r}\csc^2(1 + \ln r)\,dr$
8. $\int \frac{\cos(1 - \ln v)}{v}\,dv$

Summary

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Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x > a > 0)$

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Integration by Parts

Integration by parts is a technique for simplifying integrals of the functions $\int f(x)g(x)\,dx$.

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

Examples

$\int x \cos x\,dx$ $\quad f(x) = x$ and $g(x) = \cos x$

$\int x^2 e^x\,dx$ $\quad f(x) = x^2$, $g(x) = e^x$

$\int \ln x\,dx$ $\quad f(x) = \ln x$ and $g(x) = 1$

$\int e^x \cos x\,dx.$ $\quad f(x) = e^x$ $\quad g(x) = \cos x$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Find $\int x \cos x \, dx$.

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = x, \quad dv = \cos x \, dx,$$ Simplest antiderivative of $\cos x$

$$du = dx, \quad v = \sin x.$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

EXAMPLE 2 Find $\int \ln x \, dx$.

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx, \qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C$$

EXAMPLE 3 Evaluate $\int x^2 e^x\, dx$.

Solution With $u = x^2$, $dv = e^x\, dx$, $du = 2x\, dx$, and $v = e^x$, we have

$$\int x^2 e^x\, dx = x^2 e^x - 2\int x e^x\, dx.$$

we integrate by parts again with $u = x$, $dv = e^x\, dx$. Then $du = dx$, $v = e^x$, and

$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C.$ Using this last evaluation, we then obtain

$$\int x^2 e^x\, dx = x^2 e^x - 2\int xe^x\, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

The technique of Example 3 works for any integral $\int x^n e^x\, dx$ in which n is a positive integer,

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EXAMPLE 4 Evaluate $\int e^x \cos x \, dx.$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

we use integration by parts with

$u = e^x, \quad dv = \sin x \, dx, \quad v = -\cos x, \quad du = e^x \, dx.$ Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

EXAMPLE 5 Obtain a formula that expresses the integral $\int \cos^n x\, dx$ in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx, \qquad \text{so that}$$

$$du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$$ Integration by parts then gives

$$\begin{aligned}
\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.
\end{aligned}$$

If we add $(n-1)\int \cos^n x\, dx$ to both sides of this equation, we obtain

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

$$\begin{aligned}
\int \cos^3 x\, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx \\
&= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.
\end{aligned}$$

EXAMPLE Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \left|\sec x + \tan x\right| + C.$$

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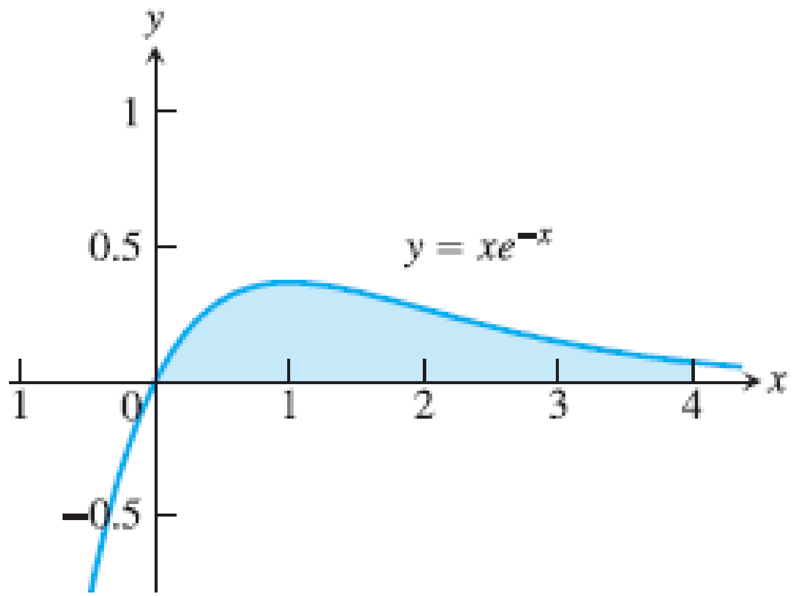
EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure . Its area is $\int_0^4 xe^{-x}\,dx$.

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx$$



$$= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4$$

$$= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome; or, you choose substitutions for a repeated integration by parts that just ends up giving back the original integral you were trying to find. In situations like these, there is a way to organize the calculations that prevents these pitfalls and makes the work much easier. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

$$\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$$

EXAMPLE 8 Evaluate $\int x^3 \sin x \, dx$.

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Exercises

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$

3. $\int t^2 \cos t\, dt$

5. $\int_1^2 x \ln x\, dx$

7. $\int xe^x\, dx$

9. $\int x^2 e^{-x} dx$

11. $\int \tan^{-1} y\, dy$

2. $\int \theta \cos \pi\theta\, d\theta$

4. $\int x^2 \sin x\, dx$

6. $\int_1^e x^3 \ln x\, dx$

8. $\int xe^{3x}\, dx$

10. $\int (x^2 - 2x + 1)\, e^{2x}\, dx$

13. $\int x \sec^2 x\, dx$

12. $\int \sin^{-1} y\, dy$

14. $\int 4x \sec^2 2x\, dx$

15. $\int x^3 e^x\, dx$

17. $\int (x^2 - 5x)e^x\, dx$

19. $\int x^5 e^x\, dx$

21. $\int e^\theta \sin \theta\, d\theta$

23. $\int e^{2x} \cos 3x\, dx$

16. $\int p^4 e^{-p}\, dp$

18. $\int (r^2 + r + 1)e^r\, dr$

20. $\int t^2 e^{4t}\, dt$

22. $\int e^{-y} \cos y\, dy$

24. $\int e^{-2x} \sin 2x\, dx$

THANKS FOR YOUR ATTENTION