CIHAN UNIVERSITY
SULAIMANIYA

Mathematics-1-

For Engineering

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Chapter 3
Transcendental Functions

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Transcendental Functions

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OVERVIEW

Functions can be classified into two broad complementary groups called

- *algebraic functions* and
- *transcendental functions*

In this chapter we investigate the calculus of important transcendental functions, including :

- *the logarithmic,*
- *exponential,*
- *inverse trigonometric, and*
- *hyperbolic functions*.

One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain.

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE 1

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y-value for at least two different x-values and is therefore not one-to-one

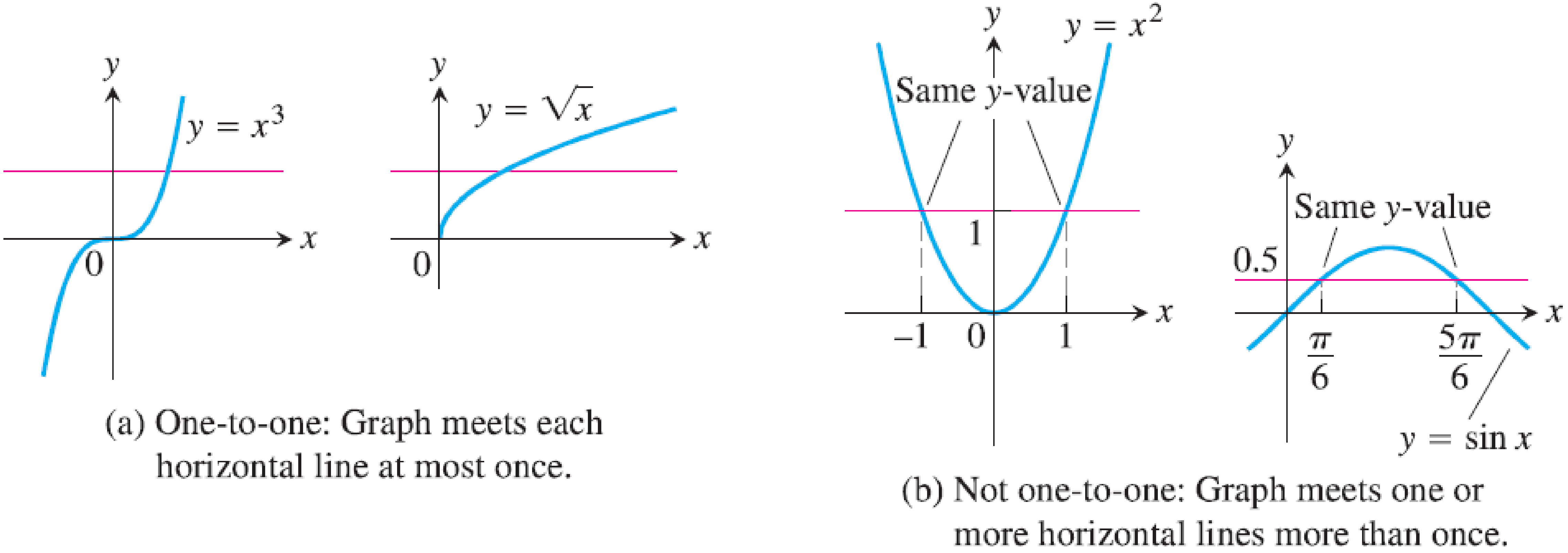


FIGURE 1 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

Inverse Functions

*An **inverse function** or an anti function is defined as a function, which can reverse into another function. In simple words, if any function "f" takes x to y then, the inverse of "f" will take y to x. If the function is denoted by 'f' or 'F', then the inverse function is denoted by f^{-1} or F^{-1}.*

The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

EXAMPLE 2 Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

Only a one-to-one function can have an inverse

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f\text{)}$$

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:* $y = \frac{1}{2}x + 1$ then $2y = x + 2$ and
$x = 2y - 2.$
2. *Interchange x and y:* $y = 2x - 2.$

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2.$

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

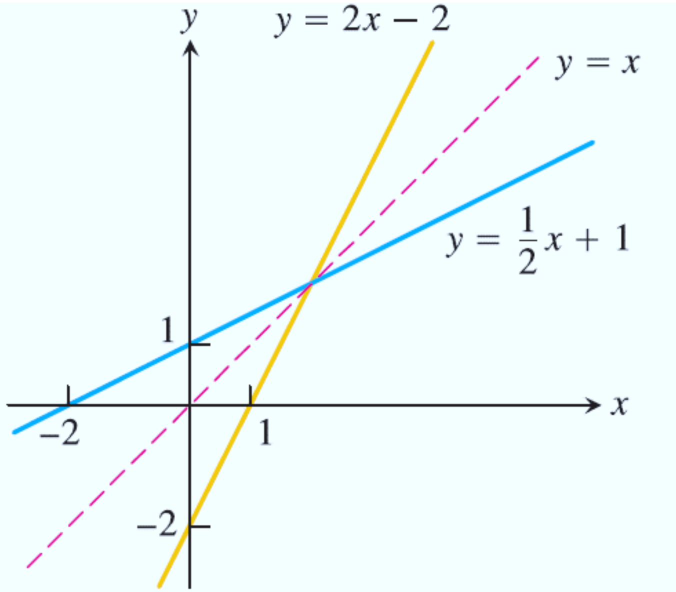


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

3.1 EXPONENTIAL FUNCTION

An exponential function is defined by

$$f(x) = a^x, \qquad \text{so that} \quad x = \log_a f, \qquad a > 0,$$

where a is the **base** and x is the **index**.

The most useful exponential function is $f(x) = e^x \equiv \exp x$ where $e = 2.71828$.

EXAMPLES

1. If $8 = x^3$ then $x = 8^{1/3} = 2$.
2. If $3 = \log_2 y$ then $y = 2^3 = 8$.
3. If $2 = \log_{10} y$ then $y = 10^2 = 100$.
4. If $y = \log_2 16$ then since $16 = 2^4$, $y = 4$.

Exponential function

Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in many natural and industrial situations. The variety of models based on these functions partly accounts for their importance.

Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time I. Examples of such quantities include the amount of a decaying radioactive material, the size of a population, and the temperature difference between a hot object and its surrounding medium. Such quantities are said to undergo **exponential** change.

3.2 INDEX LAWS

1. $a^i = \underbrace{a.a\ldots a}_{i \text{ times}}$, for i an integer.
2. $a^m a^n = a^{m+n}$ } Equal Bases Rule
3. $\dfrac{a^m}{a^n} = a^{m-n}$ } Equal Bases Rule
4. $a^m b^m = (ab)^m$ } Equal Indices Rule
5. $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$ } Equal Indices Rule
6. $a^{-n} = \dfrac{1}{a^n}$
7. $(a^m)^n = a^{mn}$ Power of a Power Rule
8. $a^0 = 1$

3.3 LOGARITHM RULES

1. $\log_a(xy) = \log_a x + \log_a y$ — Log of a Product
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ — Log of a Quotient
3. $\log_a x^p = p\log_a x$ — Log of a Power
4. $\log_a(a^x) = x$
5. $a^{\log_a x} = x$
6. $\log_a 1 = 0$ and $\log_a a = 1$

The natural log, or ln, is the inverse of *e*. The letter '*e*' represents a mathematical constant also known as the natural exponent. Like π, *e* is a mathematical constant and has a set value. The value of *e* is equal to approximately 2.71828.

Product Rule

- **ln (x)(y) = ln (x) + ln (y)**
- The natural log of the multiplication of x and y is the sum of the ln of x and ln of y.
- Example: ln (8)(6) = ln (8) + ln (6)

Quotient Rule

- **ln (x/y) = ln (x) - ln (y)**
- The natural log of the division of x and y is the difference of the ln of x and ln of y.
- Example: ln (7/4) = ln (7) - ln (4)

Reciprocal Rule

- **ln (1/x) = −ln (x)**
- The natural log of the reciprocal of x is the opposite of the ln of x.
- Example: ln (⅓)= -ln (3)

Power Rule

- **ln (x^y) = y * ln (x)**
- The natural log of x raised to the power of y is y times the ln of x.
- Example: ln (5^2) = 2 * ln (5)

Angles

Angles are measured in degrees or radians.

$$s = r\theta \qquad (\theta \text{ in radians}).$$

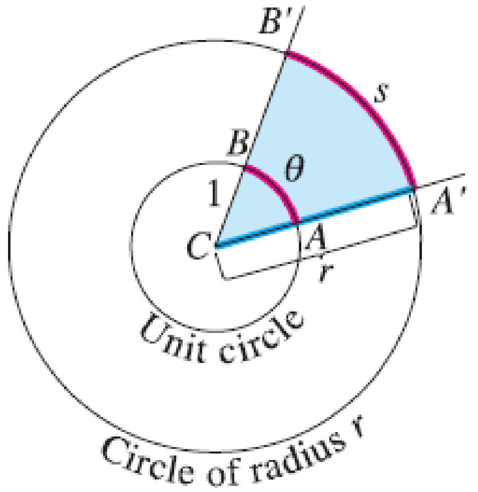


FIGURE The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

If the circle is a unit circle having radius $r = 1$,

$$\pi \text{ radians} = 180° \qquad \text{and} \qquad 1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees}$$

$$\text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

TABLE 1 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

The Six Basic Trigonometric Functions

We define the trigonometric functions in terms of the coordinates of the point $P(x, y)$

sine: $\sin\theta = \dfrac{y}{r}$ **cosecant:** $\csc\theta = \dfrac{r}{y}$

cosine: $\cos\theta = \dfrac{x}{r}$ **secant:** $\sec\theta = \dfrac{r}{x}$

tangent: $\tan\theta = \dfrac{y}{x}$ **cotangent:** $\cot\theta = \dfrac{x}{y}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

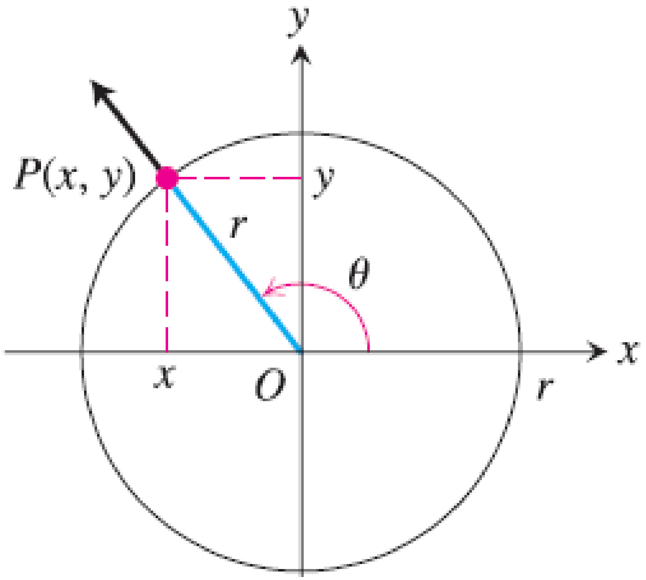


FIGURE
The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

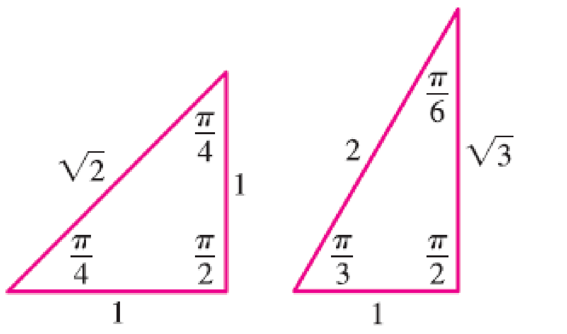


FIGURE Radian angles and side lengths of two common triangles.

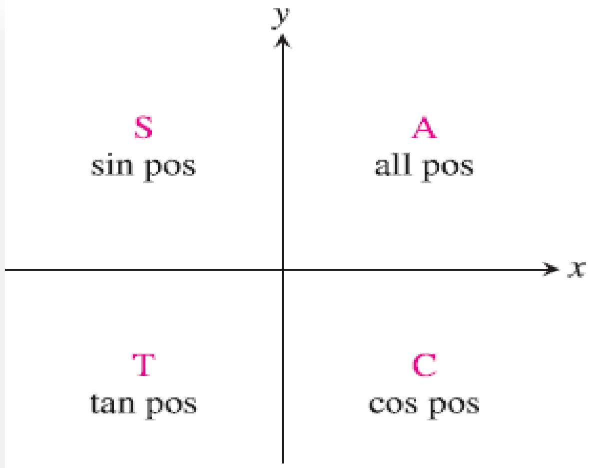
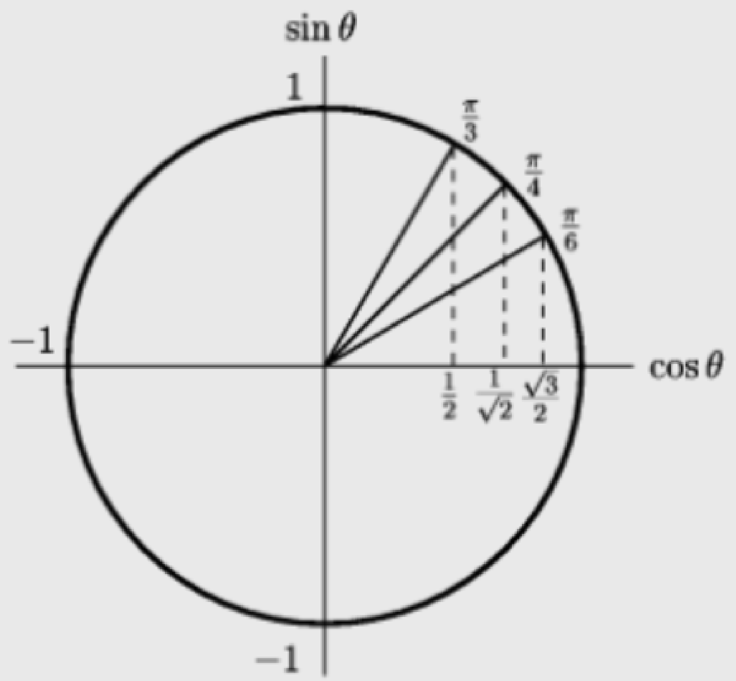


FIGURE The CAST rule,

3.4 TRIGONOMETRIC FUNCTIONS

The unit circle can be used as an aid for finding the sin and cos of common angles. For example, $\cos \pi/6 = \sqrt{3}/2$. By symmetry all the other major angles can be found.



DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

EXAMPLES

$\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + 2\pi) = \tan\theta$,

Similarly, $\cos(\theta - 2\pi) = \cos\theta$,

$\sin(\theta - 2\pi) = \sin\theta$,

Periods of Trigonometric Functions

Period π :

$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

Period 2π :

$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

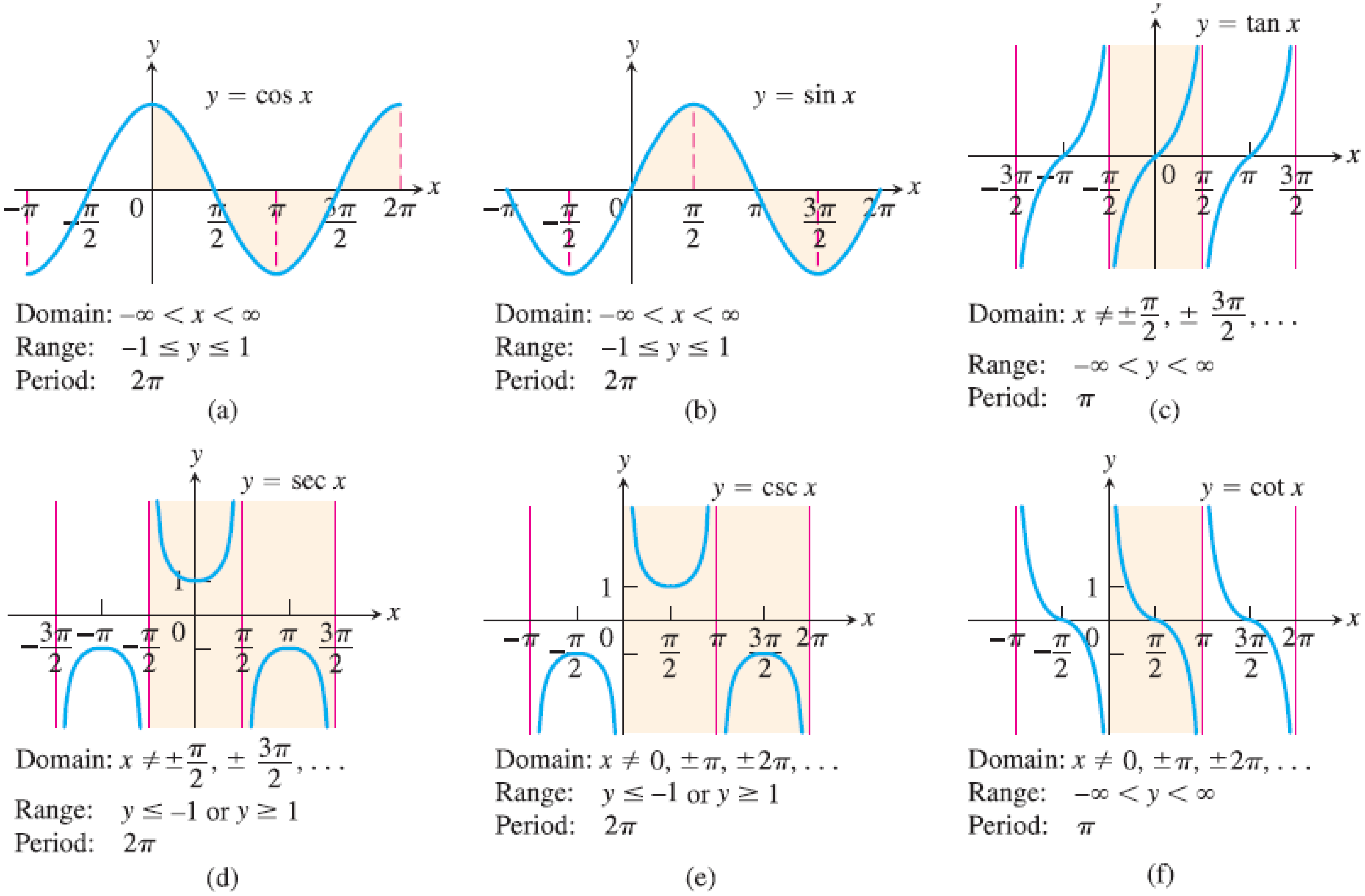


FIGURE Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Even

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x$$

Odd

$$\sin(-x) = -\sin x$$
$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\cot(-x) = -\cot x$$

Trigonometric Identities

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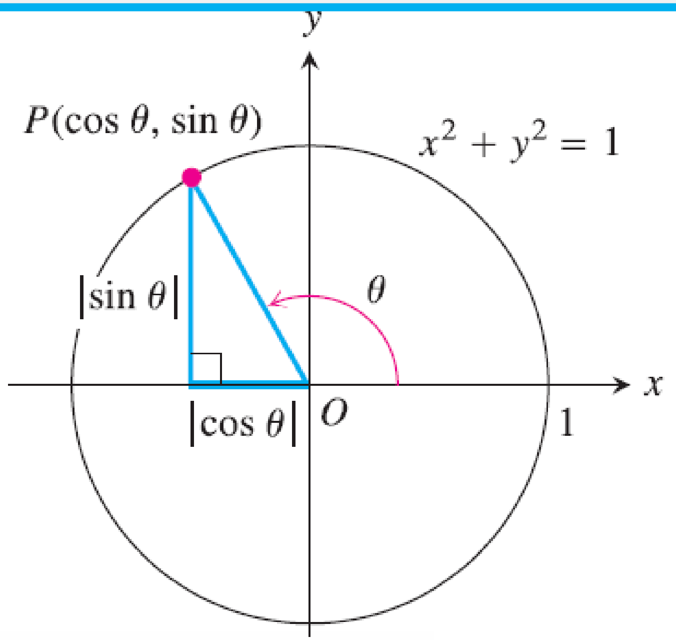
The coordinates of any point $P(x, y)$

$x/r = \cos\theta \Rightarrow x = r\cos\theta,$

and $y/r = \sin\theta \Rightarrow y = r\sin\theta.$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle

$$\cos^2\theta + \sin^2\theta = 1. \quad \text{.................1}$$



Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

Double-Angle Formulas

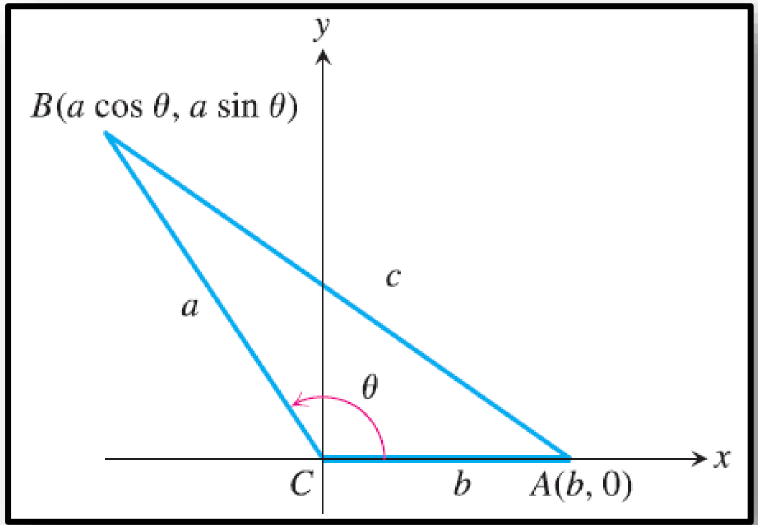
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$



The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta .$$

Inverse Trigonometric Functions

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Defining the Inverses

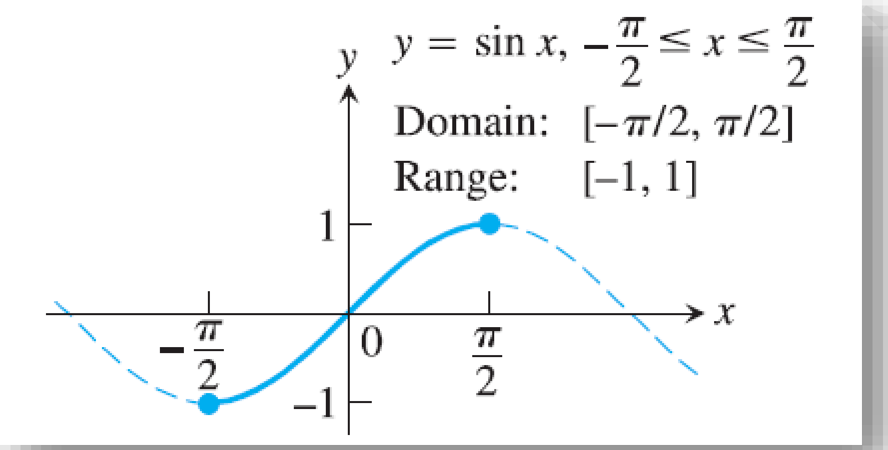
The six basic trigonometric functions are not one-To-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-To-one

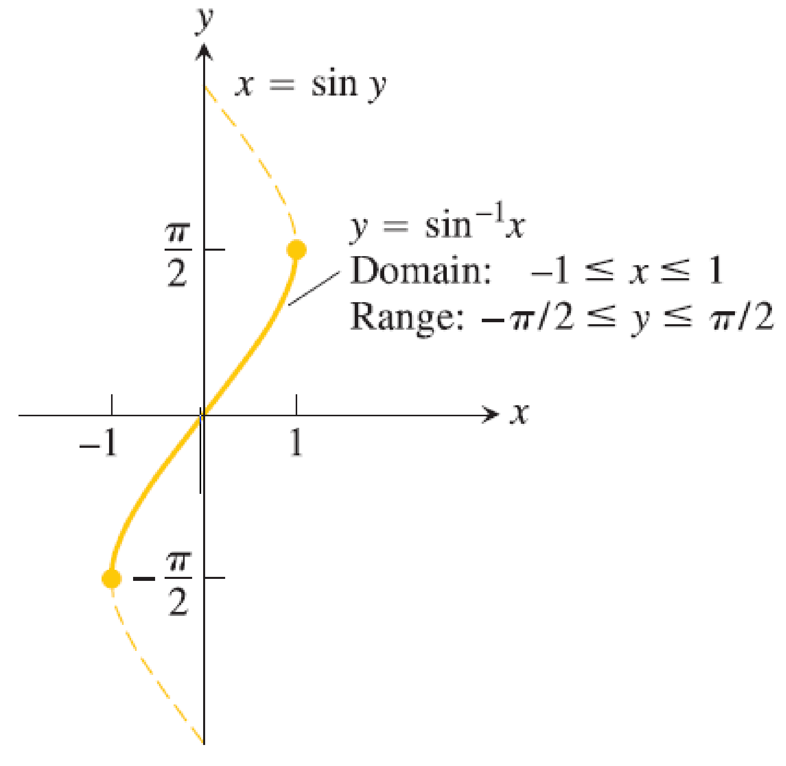
DEFINITION

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$, we make it one-to-one, so that it has an inverse $\sin^{-1} x$ Similar domain restrictions can be applied to all six trigonometric functions.





$y = \sin^{-1} x$ or $y = \arcsin x$

EXAMPLE Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution **(a)** We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \quad \text{because } \sin(\pi/3) = \sqrt{3}/2 \text{ and } \pi/3$$

belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function.

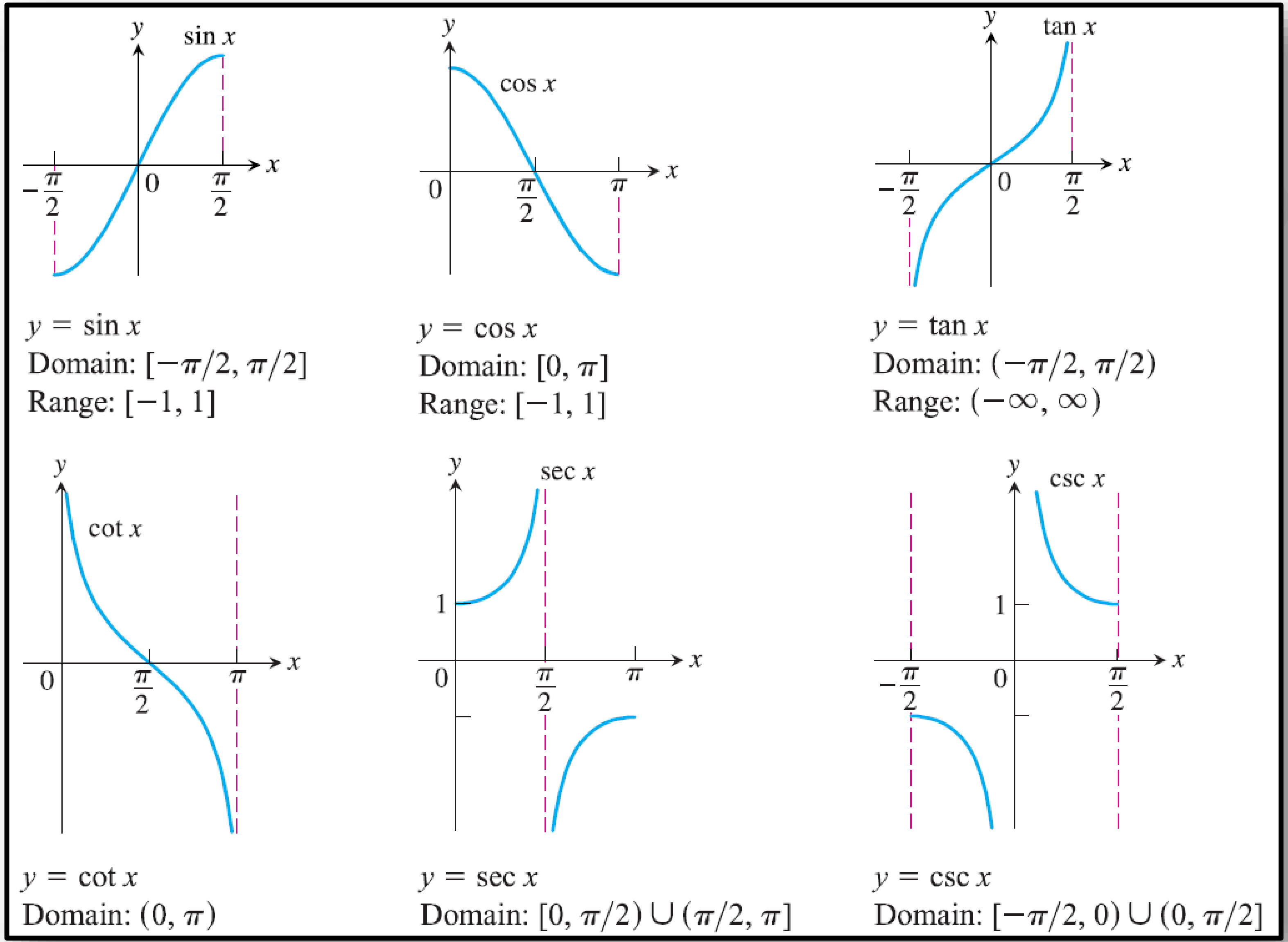
(b) We have $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function.

DEFINITION

$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.



Caution **The - 1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of sin *x* is (sinx)-I = l/sinx = cscx.**

$y = \sin^{-1} x$	or	$y = \arcsin x$
$y = \cos^{-1} x$	or	$y = \arccos x$
$y = \tan^{-1} x$	or	$y = \arctan x$
$y = \cot^{-1} x$	or	$y = \operatorname{arccot} x$
$y = \sec^{-1} x$	or	$y = \operatorname{arcsec} x$
$y = \csc^{-1} x$	or	$y = \operatorname{arccsc} x$

3.6 HYPERBOLIC FUNCTIONS

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}.

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Definitions and Identities

1. $$2\sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x.$$

2. $$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

3. $$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

4. $$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

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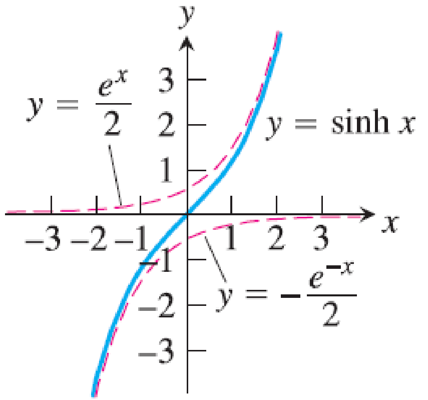
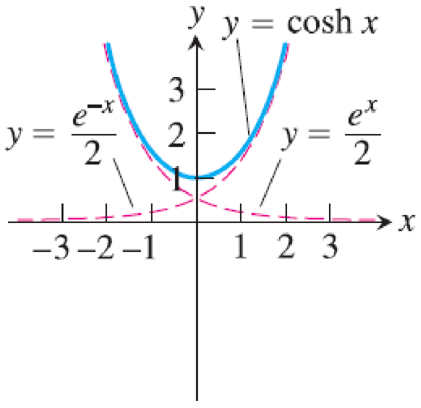
GRAPH OF THE HYPORBELIC FUNCTION



(a)

Hyperbolic sine:

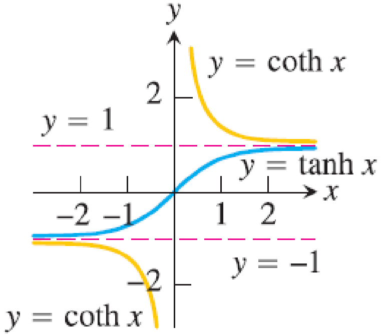
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

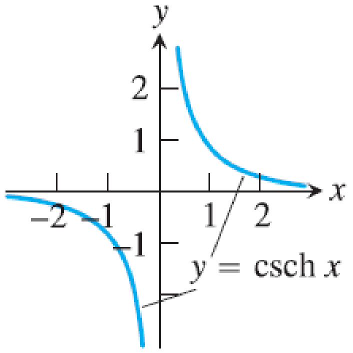
$$\cosh x = \frac{e^x + e^{-x}}{2}$$



(c)

Hyperbolic tangent:

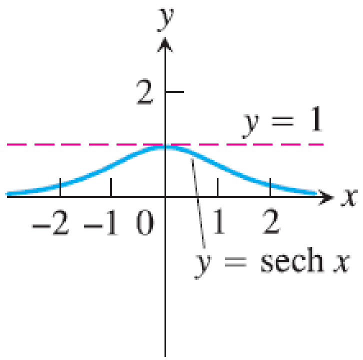
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

1. Simplify as much as possible

(i) $6x^3y^{-2} \times \frac{1}{24}x^{-5}y^4$

(ii) $8^{-\frac{2}{3}}$

(iii) $2\log_{10} 5 + \log_{10} 8 - \log_{10} 2$

(iv) $3^{-\log_3 P}$

(v) $\ln x^2 + \ln y - \ln x - \ln y^2$

(vi) $e^{2\ln x}$

5. Evaluate

(i) $\tan(\pi)$

(ii) $\sin\left(\frac{6\pi}{8}\right)$

(iii) $\cos\left(\frac{11\pi}{6}\right)$

(iv) $\sec\left(\frac{4\pi}{3}\right)$

6. Simplify

(i) $\frac{1}{\cos^2\theta} - \tan^2\theta$

(ii) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

(iii) $\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$

2. Solve for t using natural logarithms:

(i) $5^t = 7$

(ii) $2 = (1.02)^t$

(iii) $3^t 7 = 2^t 5$

(iv) $Q = Q_0 a^{nt}$

(v) $y = 3 - 2\ln t$

(vi) $3y = 1 + 2e^{4t}$

3. If $\ln s = 2$ and $\ln t = 3$ calculate

(i) $\ln(st)$

(ii) $\ln(st^2)$

(iii) $\ln(\sqrt{st})$

(iv) $\ln \frac{s}{t}$

(v) $\ln \frac{s}{t^3}$

4. If $x = \ln 3$ and $y = \ln 5$ then find

(i) $e^x e^y$

(ii) e^{x+y}

(iii) e^{2x}

(iv) $e^x + e^y$

7. Solve the following for values of θ between 0 and 2π

(i) $\cos^2\theta + 3\sin^2\theta = 2$

(ii) $2\cos^2\theta = 3\sin\theta$

8. Prove the following identities:

(i) $\dfrac{1+\sin\theta}{1-\sin\theta} = (\sec\theta + \tan\theta)^2$

(ii) $3\sin^2\theta - 2 = 1 - 3\cos^2\theta$

(iii) $\sinh x - \cosh x = -e^{-x}$

(iv) $\sinh x + \cosh x = e^{x}$

9. Use the trigonometric addition of angle formulae to show

$$\cos\frac{\pi}{12} = \frac{1}{4}(\sqrt{6}+\sqrt{2}).$$

10. For the following angles find $\cos\theta$, $\sin\theta$, $\tan\theta$, and $\sec\theta$:

(i) $\theta = \dfrac{\pi}{4}$ (ii) $\theta = 13\dfrac{\pi}{6}$ (iii) $\theta = \dfrac{2\pi}{3}$

11. Use the multiple angle formulae to find $\cos\dfrac{\pi}{12}$.

13. Is $f(x) = x\cos x$ an odd or even function?

12. In an experiment you have to calculate the time to melt a block of ice using the formula

$$t = \frac{l\,(\lambda\rho - cT_0\,\rho)}{h\,T_a}$$ where

$l = 0.1,\quad \lambda = 3\times10^5,$
$c = 2\times10^3,\quad T_0 = -20,$

AND

$T_a = 20,\quad h = 10,$
$\rho = 1\times10^3.$

find t

THANKS FOR YOUR ATTENTION